18.600 Recitation 10

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Partial solutions available at math.mit.edu/~visheshj

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Solution Problem 1. (a) We have

$$H = \frac{X_1 + \cdots + X_n}{n},$$

where $X_1, \ldots, X_n$ are i.i.d. samples with the same distribution as X. Therefore, $\mathrm{E}[H] = \mathrm{E}[X] = h$ (by linearity of expectation), whereas by independence of $X_1, \ldots, X_n$,

$$\mathrm{Var}(H) = \frac{1}{n^2}\mathrm{Var}(X_1 + \cdots + X_n) = \frac{\mathrm{Var}(X)}{n}.$$

Hence, $\sigma(H) = \frac{1.5}{\sqrt{n}}$.

(b) From the previous part, we want to find the minimum value of n such that $1.5/\sqrt{n} < 0.01$. This means $\sqrt{n} > 150$ i.e. $n > 22500$. So, the minimum n is 22501.

(c) From Chebyshev's inequality, we know that

$$\Pr(|H - h| \geq c) \leq \frac{\sigma(H)^2}{c^2} \leq \frac{2.25}{c^2 n}.$$

We would like to take $c = 0.05$ and the right hand side to be ≤ 0.01. Plugging this in, we get

$$\frac{2.25}{(0.05)^2 n} \leq 0.01$$

i.e. the minimum is $n = 225/(0.05)^2 = 90000$.

(d) More generally, let us prove the following: if X is a random variable supported in $[m, M]$, then $\sigma[X] \leq \frac{(M-m)}{2}$. To see this, note that

$$\begin{aligned} f(t) := \mathrm{E}[(X-t)^2] &= \mathrm{E}[((X-\mu) + (\mu - t))^2] \\ &= \mathrm{Var}[X] + (\mu - t)^2 \end{aligned}$$

and hence is minimized at $t = \mu$. Therefore, we get

$$\begin{aligned} \mathrm{Var}[X] &\leq f((m+M)/2) \\ &= \mathrm{E}[(X - (m+M)/2)^2] \\ &= \frac{1}{4}\mathrm{E}\left[\{(X-m) + (X-M)\}^2\right]. \end{aligned}$$

Note that since $X - m \geq 0$ and $X - M \leq 0$, we have $\{(X-m) + (X-M)\}^2 \leq \{(X-m) - (X-M)\}^2 = (M-m)^2$, which gives the desired conclusion.

Solution Problem 2. **(a)** It is clear that the MGF does not exist for $|t| \geq 1$. Let $|t| < 1$. Then,

$$\begin{aligned}
\mathrm{E}[e^{tX}] &= \frac{1}{2}\int_{-\infty}^{\infty} e^{tx}e^{-|x|}dx \\
&= \frac{1}{2}\int_{0}^{\infty} e^{(t-1)x}dx + \frac{1}{2}\int_{-\infty}^{0} e^{(t+1)x}dx \\
&= \frac{1}{2}\left(-\frac{1}{(t-1)} + \frac{1}{(t+1)}\right) \\
&= \left(\frac{1}{1-t^2}\right) \\
&= 1 + t^2 + t^4 + t^6 + \dots.
\end{aligned}$$

From this, we see that $\mathrm{E}[X] = 0$ (this is also clear by symmetry) and $\mathrm{Var}[X] = \mathrm{E}[X^2] = 2$.

(b) We know that the MGF of $\mathcal{N}(\mu, \sigma^2) = \exp\left(\frac{\sigma^2 t^2}{2} + \mu t\right)$, from which we see that the required MGF is $\exp\left(t^2\right)$.

(c) By independence, we have

$$\begin{aligned}
\mathrm{E}[e^{tS_n}] &= \mathrm{E}\left[\exp(tX_1/\sqrt{n} + \cdots + tX_n/\sqrt{n})\right] \\
&= \mathrm{E}\left[\prod_{i=1}^{n} \exp(tX_i/\sqrt{n})\right] \\
&= \prod_{i=1}^{n} \mathrm{E}\left[\exp(tX_i/\sqrt{n})\right] \\
&= \prod_{i=1}^{n} \left(\frac{1}{1-(t/\sqrt{n})^2}\right) \\
&= \left(1 - \frac{t^2}{n}\right)^{-n}.
\end{aligned}$$

As $n \to \infty$, this converges to $\exp(t^2)$.

Solution Problem 3. Yes. Recall the derivation of Chebyshev's inequality:

$$\Pr(|X - \mathrm{E}[X]| \geq c) = \Pr\left(|X - \mathrm{E}[X]|^2 \geq c^2\right) \leq \mathrm{E}[(X - \mathrm{E}[X])^2]/c^2 = \sigma^2/c^2.$$

This shows that the inequality is tight if and only if

$$c^2 \Pr\left(|X - \mathrm{E}[X]|^2 \geq c^2\right) = \mathrm{E}[(X - \mathrm{E}[X])^2] = \mathrm{Var}(X).$$

Let $X = \mu + c$ with probability p, $X = \mu - c$ with probability p, and $X = \mu$ with probability $1 - 2p$, for p to be decided below. Then, we see that $\mathrm{E}[X] = \mu$ and $\mathrm{Var}[X] = 2pc^2 = c^2 \Pr(|X - \mathrm{E}[X]|^2 \geq c^2)$. So, if we want the variance to be equal to σ^2, we must take $p = \frac{\sigma^2}{2c^2}$, which makes sense since $c \geq \sigma$.