18.600 Recitation 11

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Partial solutions available at math.mit.edu/∼visheshj

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Solution Problem 1. (a) The number of undergrads who enroll in the class is distributed as $\text{Bin}(4000, 0.05)$, which we approximate by $\mathcal{N}(4000\times 0.05, 4000\times 0.05\times 0.95) = \mathcal{N}(200, 190) \sim 200 + \sqrt{190}\mathcal{N}(0,1)$. Therefore, the probability that 250 or more students enroll is approximately the probability that $\mathcal{N}(0,1) \geq 50/\sqrt{190}$, which is $1 - \Phi(50/\sqrt{190}) \approx 0.00014$.

(b) Let N denote the number of undergrads who enroll in the class. By Markov's inequality, we have $\Pr[N \geq 250] \leq \text{E}[N]/250 = 200/250 = 0.8$.

(c) We have, for any $t > 0$, that $\Pr[N \geq 250] = \Pr[\exp(tN) \geq \exp(250t)] \leq e^{-250t}M_N(t) = e^{-250t}(0.05e^t + 0.95)^{4000}$. Substituting $t = 0.2$, this is ≈ 0.00257.

Solution Problem 2. (a) Note that the transition matrix is given by

$$P = \begin{bmatrix} 0 & 1-p & 0 & p \\ p & 0 & 1-p & 0 \\ 0 & p & 0 & 1-p \\ 1-p & 0 & p & 0 \end{bmatrix} \tag{1}$$

where p_{ij} is the probability of transitioning from state i to state j. Therefore, the distribution of X_n, starting from $X_0 = 1$, is given by $(1,0,0,0)P^n$.

(b) No, the Markov chain is not ergodic. If n is even, then X_n must be 1 or 3, while if n is odd, then X_n must be 2 or 4.

(c) The new transition matrix is given by

$$P = \begin{bmatrix} 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & 1/3 \\ 1/3 & 0 & 1/3 & 1/3 \end{bmatrix} \tag{2}$$

It is easy to see that this new Markov chain is ergodic. Therefore, we know that the distribution of X_k, as k goes to infinity, converges to the (unique) stationary distribution, which in our case is given by $(1/4, 1/4, 1/4, 1/4)$ (it is the unique solution to the system of equations: $2a_1 = a_2 + a_4; 2a_2 = a_1 + a_3; 2a_3 = a_2 + a_4; 2a_4 = a_1 + a_3; a_1 + a_2 + a_3 + a_4 = 1$).