

18.600 Recitation 12

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Partial solutions available at math.mit.edu/~visheshj

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Problem 1. You throw two dice. Let X be the value of the first die, let Y be the value of the second die, and let $Z = X + Y$.

- (a) Find $H(X, Y)$.
- (b) Find $H(Z)$.
- (c) Find $H_X(Z)$.
- (d) Find $H(X, Z)$. Could you have done this directly without using the previous part?

Problem 2. Bob has four coins and tosses them simultaneously; let X denote the number of heads he sees. He wants to send X to Alice encoded as a binary string, in such a way that the expected number of bits is as small as possible.

- (a) Use Shannon's Noiseless Coding Theorem to find an upper and lower bound for the expected number of bits when he uses an optimal encoding.
- (b) Find a coding scheme for which the expected number of bits is between the upper and lower bound you just found.
- (c) Now suppose that Bob repeats the experiment 100 times and wants to send the outcomes X_1, X_2, \dots, X_{100} to Alice. Use Shannon's Noiseless Coding Theorem to find an upper and lower bound for the expected number of bits per symbol when he uses an optimal encoding.

Problem 3. For two probability distributions p and q on $\{1, \dots, n\}$, define their *relative entropy* by

$$\mathbf{KL}(p||q) := \sum_{i=1}^n p_i \log_2(p_i/q_i).$$

For the problems below, we will use Jensen's inequality, which says that for any convex function f and any random variable X , $E[f(X)] \geq f(E[X])$. Moreover, if f is strictly convex, then equality holds if and only if X is a constant.

- (a) Show that $\mathbf{KL}(p||q) \geq 0$ for any probability distributions p and q , and equality holds if and only if $p = q$. *Hint:* The function $x \mapsto -\log(x)$ is strictly convex.
- (b) Use this to show that $H(p) \leq \log_2(n)$ for any probability distribution p on $\{1, \dots, n\}$, with equality attained if and only if p is the uniform distribution on $\{1, \dots, n\}$. Should you have expected this to be true? *Hint:* Compute $\mathbf{KL}(p||\text{Unif})$, where Unif is the uniform distribution on $\{1, \dots, n\}$.

Problem 4. Let X_1, \dots, X_n be i.i.d. $\text{Ber}(p)$ random variables. Let $X = X_1 + \dots + X_n$ (hence X is distributed as $\text{Bin}(n, p)$). Using the proof of the Chernoff bound, show that for any $\delta \in [0, 1 - p]$,

$$\Pr[X \geq (p + \delta)n] \leq \exp(-n\mathbf{KL}(\text{Ber}(p + \delta)||\text{Ber}(p))).$$