18.600 Recitation 1 - partial solutions

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In recitation, we did the first three problems in detail. We also got through problems 5(a) and 5(b), and I spoke to many of you about problem 4 as well. Here, I am giving brief solutions for problems 4 and 5. Please ask if anything is unclear.

Solution Problem 4.

There are 12^3 possible combinations of birthday months, which are captured by the following set $S = \{(1,1,1),(1,1,2),\dots,(12,12,12)\}$. Let E be those combinations in which at least two people have the same birthday month. It's easier to count the combinations in E^c i.e. those combinations in which all three people have different birthday months.

Indeed, for a combination to be in E^c, there are 12 choices for the first person's birthday month, followed by 11 choices for the second person's birthday month (it can be any month which is not the first person's birthday month), followed by 10 choices for the third person's birthday month (it can be any month which is not the first or the second person's birthday month, and we know that the first two people take up two months). Thus, there are $12 \times 11 \times 10$ combinations in E^c.

Hence, $P(E) = 1 - P(E^c) = 1 - \frac{|E^c|}{|S|} = 1 - \frac{12\cdot 11\cdot 10}{12^3} = \frac{17}{72}$.

Solution Problem 5. (a) Use "stars and bars" (Theorem one in the link). In our case, there are 20 stars and 3 bars. Since we want a solution in *positive* integers, there are 19 locations for where to place the 3 bars (we cannot place a bar on the left of all the stars or on the right of all the stars, and we also cannot place two consecutive bars without a star in between). This can be done in $\binom{19}{3}$ ways. Please convince yourself that each such way leads to a different solution, and every solution is captured by this process.

(b) **Method 1**. Use the following trick to reduce to the previous problem: set $y_1 = x_1+1,\dots,y_4 = x_4+1$. Then, $y_1+y_2+y_3+y_4 = 20$, and y_1,y_2,y_3,y_4 are positive integers. Therefore, as above, there are $\binom{19}{3}$ solutions for (y_1,y_2,y_3,y_4). Since any such solution leads to a corresponding solution for (x_1,x_2,x_3,x_4) and vice versa (do you see why?), there are $\binom{19}{3}$ solutions to our problem.

Method 2. Use "stars and bars" (Theorem two in the link) directly. We have 16 stars and 3 bars. Notice that *any* arrangement of the stars and bars leads to a unique valid solution, and vice versa (do you see the difference between this setting and part (a)?). Therefore, there are $\binom{19}{3}$ solutions.

(c) **Method 1**. Reduce to the previous problem as follows: every distinct solution $x_1+x_2+x_3+x_4 = 16$, where x_1,x_2,x_3,x_4 are non-negative integers leads to a unique non-negative integral solution of $x_1+x_2+x_3 \leq 16$. Conversely, any solution of the latter leads to a unique solution of the former (do you see why?). Therefore, the number of solutions is equal to the answer in part (b), which is $\binom{19}{3}$.

Method 2. By writing the inequality as a combination of equalities $x_1+x_2+x_3 = 16$, $x_1+x_2+x_3 = 15$, $\dots$, $x_1+x_2+x_3 = 0$, it follows from part (a) that the number of solutions is

$$\binom{18}{2} + \binom{17}{2} + \cdots + \binom{2}{2}.$$

We will simplify the sum using the following useful identity (exercise, prove this!)

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1},$$

from which we see that

$$\binom{18}{2} = \binom{19}{3} - \binom{18}{3},$$
$$\binom{17}{2} = \binom{18}{3} - \binom{17}{3},$$

and so on, until

$$\binom{2}{2} = \binom{3}{3} - 0.$$

Adding all these together, we see that the desired sum is $\binom{19}{3}$, as before.