18.600 Recitation 2 - partial solutions
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Solution Problem 1. Note that $P(A) = P(B) = P(C) = 1/2$. Also, $A \cap B = B \cap C = C \cap A$ is the event that the card is hearts, so that $P(A \cap B) = P(B \cap C) = P(C \cap A) = 1/4$. This shows that the events are pairwise independent. However, $A \cap B \cap C$ is still the event that the card is hearts, so that $P(A \cap B \cap C) = 1/4$ and not $1/8$. This shows that the events are not mutually independent.

Solution Problem 2. The probability that a student does not return to their own room is $1 - (1/n)$. Since the events that different students do not return to their rooms are independent by definition, the desired probability is $(1 - (1/n))^n$.

Solution Problem 3. Note that if you spin the barrel, there is a $2/6 = 1/3$ chance that it will land at a bullet and you will be shot. Hence, there is a $2/3$ chance of survival.

For the other case, let B be the event that the shot already fired is a blank, and S be the event that you survive in the next shot. Hence, we are interested in $P(S|B)$. By Bayes' rule, we have $P(S|B) = P(S \cap B)/P(B) = (3/6)/(4/6) = 3/4$. Hence, the chance of survival is greater in this case.

Solution Problem 4. Let A denote the event of a person having an accident. Let H denote the event of a person being high risk.

(a) By the law of total probability, we have $P(A) = .5 \times .2 + .15 \times .3 + .05 \times .5 = 0.17$.

(b) We are interested in the event $H|A$. By Bayes' rule, $P(H|A) = P(A|H)P(H)/P(A) = (.5 \times .2)/.17 = 10/17$.

Solution Problem 5. Let M be the event of a person being male, F be the event of a person being female and C be the event of a person being colorblind. We are interested in the event $M|C$. By Bayes' rule, we have $P(M|C) = P(C|M)P(M)/P(C) = 0.8 \times P(M)/P(C)$.

In case the number of males and the number of females is the same, we have $P(M) = P(F) = 0.5$, and by the law of total probability, $P(C) = 0.5 \times P(C|M) + 0.5 \times P(C|F) = 0.425$. Hence, we get $P(M|C) = 0.8 \times 0.5/0.425 = 16/17$

In case the number of males is twice the number of females, we have $P(M) = 2/3$ and $P(F) = 1/3$. Again, by the law of total probability, we have $P(C) = \frac{2}{3} \times P(C|M) + \frac{1}{3}P(C|F) = 0.55$. Hence, we get $P(M|C) = (0.8 \times \frac{2}{3})/0.55 = 32/33$.

Solution Optimal Problem. Note that (please convince yourself!) this problem is equivalent to the following: the hat checker randomly permutes all the hats once at the start, and then hands the ith hat in the permutation to the ith customer who arrives. Also convince yourself that since the permutation is chosen randomly from among all permutations, the order in which the customers arrive does not matter. So, we may assume that the customers arrive in some fixed order, say customer 1 followed by customer 2 and so on. Thus, the problem is equivalent to the following: what fraction of the $n!$ permutations of the numbers $1, \ldots, n$ leaves no number in its original position? Such permutations are called *derangements*. Let's denote the set of derangements of $1, \ldots, n$ by D_n, so that the solution to our problem is $|D_n|/n!$. We will now compute $|D_n|$.

For this, we will use the principle of inclusion-exclusion. Let P_i denote the set of all permutations which leave i fixed. Hence, $D_n = P_1^c \cap \cdots \cap P_n^c$ i.e. $D_n^c = P_1 \cup \cdots \cup P^n$. Hence,

$$|D_n^c| = \sum_i |P_i| - \sum_{i \neq j} |P_i \cap P_j| + \sum_{i,j,k \text{ distinct}} |P_i \cap P_j \cap P_k| - \ldots .$$

The kth term in this sequence is

$$(-1)^{k-1} \sum_{i_1,\ldots,i_k \text{ distinct}} |P_{i_1} \cap \cdots \cap P_{i_k}| = (-1)^{k-1} \binom{n}{k} (n-k)!,$$

since there are $\binom{n}{k}$ ways to choose k distinct $i_1, \ldots, i_k$, and there are $(n-k)!$ permutations fixing these numbers. Hence, we see that

$$|D_n^c| = -\sum_{k=1}^n (-1)^k \binom{n}{k} (n-k)! = -\sum_{k=1}^n (-1)^k \frac{n!}{k!},$$

so that

$$|D_n| = n! - |D_n^c| = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \right).$$