18.600 Recitation 3

Recitation Instructor: Vishesh Jain

math.mit.edu/∼visheshj

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Solution Problem 1. Let's start by computing the pmf of X. For $0 \le i \le 4$, we have

$$\mathbb{P}[X = i] = \frac{\binom{4}{i}}{16}.$$

Therefore,

$$\begin{aligned}
\mathbb{E}[Y] &= \sum_{i=0}^{4} \left(i(i-2)^2(i-4)\right) \frac{\binom{4}{i}}{16} \\
&= \left(1(1-2)^2(1-4)\right) \frac{\binom{4}{1}}{16} + \left(3(3-2)^2(3-4)\right) \frac{\binom{4}{3}}{16} \\
&= -\frac{12}{16} + -\frac{12}{16} \\
&= -\frac{3}{2}.
\end{aligned}$$

Next,

$$\begin{aligned}
\mathbb{E}\left[Y^4\right] &= \sum_{i=0}^{4} \left(i(i-2)^2(i-4)\right)^4 \frac{\binom{4}{i}}{16} \\
&= 81 \times \frac{4}{16} + 81 \times \frac{4}{16} \\
&= \frac{81}{4}.
\end{aligned}$$

Note that this is *not* equal to $\mathbb{E}[Y]^4$.

Solution Problem 2. Method 1: Direct computation using the probability mass function.

Let X_i be the random variable which is equal to 1 if the fan makes the i^{th} throw and is 0 if the fan misses the i^{th} throw. Let W be the amount of money (in dollars) won by the fan. Then,

$$W = 500(X_1 + X_2 + X_3).$$

Since the throws are independent, we have

$$\begin{aligned}
\mathbb{P}[W = 0] &= .8 \times .8 \times .5 = .32 \\
\mathbb{P}[W = 500] &= .2 \times .8 \times .5 + .8 \times .2 \times .5 + .8 \times .8 \times .5 = .48 \\
\mathbb{P}[W = 1000] &= .2 \times .2 \times .5 + .8 \times .2 \times .5 + .2 \times .8 \times .5 = .18 \\
\mathbb{P}[W = 1500] &= .2 \times .2 \times .5 = .02
\end{aligned}$$

Therefore, we get

$$\mathbb{E}[W] = .48 \times 500 + .18 \times 1000 + .02 \times 1500 = 450.$$

Method 2: Use linearity of expectation.

We have

$$\mathbb{E}[W] = 500\mathbb{E}[X_1 + X_2 + X_3]$$

$$
\begin{aligned}
&= 500\,(\mathbb{E}[X_1] + \mathbb{E}[X_2] + \mathbb{E}[X_3]) \\
&= 500\,(.2 + .2 + .5) \\
&= 500\,(.9) \\
&= 450.
\end{aligned}
$$

Solution Problem 3. Let D_1 and D_2 denote the (independent) random variables recording the outcomes of the two fair dice. Then, $X = D_1 + D_2$.

(i) To compute the pmf of X, note that there are 36 outcomes, all of which are equally probable. Moreover, for $1 \leq i \leq 6$, the number of outcomes for which $X(= D_1 + D_2) = i$ is exactly the number of solutions to the equation $x_1 + x_2 = i$, where $6 \geq x_1, x_2 \geq 1$ are integers. If $2 \leq i \leq 6$, there are $(i-1)$ such solutions. On the other hand, if $7 \leq i \leq 12$, there are $(13-i)$ such solutions. Hence, we see that

$$
\mathbb{P}\left[X = i\right] = \begin{cases} i-1 & 2 \leq i \leq 6 \\ 13-i & 7 \leq i \leq 12 \end{cases}.
$$

Therefore,

$$
\begin{aligned}
\mathbb{E}\left[X\right] &= \sum_{i=2}^{6} \frac{i(i-1)}{36} + \sum_{i=7}^{12} \frac{i(13-i)}{36} \\
&= \sum_{i=2}^{6} \frac{i^2 - i}{36} + \sum_{i=7}^{12} \frac{13i - i^2}{36} \\
&= \frac{70 + 182}{36} \\
&= 7.
\end{aligned}
$$

(ii) The problem becomes *much* easier using linearity of expectation. Note that $\mathbb{E}[D_1] = \mathbb{E}[D_2] = \frac{1}{6}\sum_{i=1}^{6} i = \frac{7}{2}$. Therefore,

$$
\begin{aligned}
\mathbb{E}[X] &= \mathbb{E}[D_1 + D_2] \\
&= \mathbb{E}[D_1] + \mathbb{E}[D_2] \\
&= 2 \times (7/2) \\
&= 7.
\end{aligned}
$$

(iii) Since we already know the pmf of X, we could directly compute

$$
\mathbb{E}[X^2] = \sum_{i=2}^{6} \frac{i^2(i-1)}{36} + \sum_{i=7}^{12} \frac{i^2(13-i)}{36} = \frac{329}{6}.
$$

As before, linearity of expectation gives an easier way out. We have,

$$
\begin{aligned}
\mathbb{E}[(D_1 + D_2)^2] &= \mathbb{E}\left[D_1^2 + D_2^2 + 2D_1D_2\right] \\
&= \mathbb{E}[D_1^2] + \mathbb{E}[D_2^2] + 2\mathbb{E}[D_1D_2] \\
&= \mathbb{E}[D_1^2] + \mathbb{E}[D_2^2] + 2\mathbb{E}[D_1]\mathbb{E}[D_2],
\end{aligned}
$$

where the last equality follows by the independence of the random variables D_1 and D_2. We already computed $\mathbb{E}[D_1] = \mathbb{E}[D_2] = (7/2)$. Moreover, we also have $\mathbb{E}[D_1^2] = \mathbb{E}[D_2^2] = \frac{1}{6}\sum_{i=1}^{6} i^2 = \frac{91}{6}$. Putting everything together, we get

$$
\mathbb{E}[X^2] = \frac{91}{6} \times 2 + 2 \times \frac{7}{2} \times \frac{7}{2}
$$

$$= \frac{91}{3} + \frac{49}{2}$$
$$= \frac{329}{6}.$$

Solution Problem 4. (i) Method 1: Direct computation using the probability mass function.

Let B denote the random variable recording the number of bad apples we have chosen. Let's compute the pmf of B. There are n^k ways to choose k apples (with replacement). The number of ways to get ℓ bad apples is $\binom{k}{\ell} b^\ell (n-b)^{k-\ell}$. Hence, we have

$$\mathbb{P}\left[B = \ell\right] = \frac{\binom{k}{\ell} b^\ell (n-b)^{k-\ell}}{n^k}.$$

Therefore,

$$\begin{aligned}
\mathbb{E}[B] &= \sum_{\ell=0}^{k} \ell \mathbb{P}\left[B = \ell\right] \\
&= \sum_{\ell=0}^{k} \frac{\ell \binom{k}{\ell} b^\ell (n-b)^{k-\ell}}{n^k} \\
&= \sum_{\ell=0}^{k} \ell \binom{k}{\ell} \left(\frac{b}{n}\right)^\ell \left(\frac{n-b}{n}\right)^{k-\ell}.
\end{aligned}$$

Here is how to sum this series: From the binomial theorem, we know that

$$(x+y)^k = \sum_{\ell=0}^{k} \binom{k}{\ell} x^\ell y^{k-\ell}.$$

Differentiating this with respect to x, we see that

$$k(x+y)^{k-1} = \sum_{\ell=1}^{k} \ell \binom{k}{\ell} x^{\ell-1} y^{k-\ell}.$$

Hence,

$$kx(x+y)^{k-1} = \sum_{\ell=0}^{k} \ell \binom{k}{\ell} x^{\ell} y^{k-\ell}.$$

Note that this is exactly the expression we got for $\mathbb{E}[B]$ with $x = \frac{b}{n}$ and $y = \frac{n-b}{n}$. Therefore, we finally get

$$\begin{aligned}
\mathbb{E}[B] &= k\frac{b}{n}\left(\frac{b}{n} + \frac{n-b}{n}\right)^{k-1} \\
&= \frac{kb}{n}.
\end{aligned}$$

Method 2: Use linearity of expectation.

Let X_i denote the random variable which is equal to 1 if the i^{th} apple we choose is bad, and is equal to 0 if the i^{th} apple we choose is good. Then, the number of bad apples we choose is equal to $X_1 + \cdots + X_k$ i.e. $B = X_1 + \cdots + X_k$ Hence, the expected number of bad apples is

$$\mathbb{E}[B] = \mathbb{E}\left[X_1 + \cdots + X_k\right] = \mathbb{E}\left[X_1\right] + \cdots + \mathbb{E}\left[X_k\right].$$

Since we are choosing the apples with replacement, $\mathbb{E}[X_i] = b/n$ for all i. Therefore,

$$\mathbb{E}[B] = \mathbb{E}[X_1 + \cdots + X_k] = \frac{kb}{n}.$$

(ii) Method 1: Direct computation using the probability mass function.

Let B denote the random variable recording the number of bad apples we have chosen. Let's compute the pmf of B. There are $\binom{n}{k}$ ways to choose k apples (without replacement). The number of ways to get ℓ bad apples is $\binom{b}{\ell}\binom{n-b}{k-\ell}$. Hence, we have

$$\mathbb{P}[B = \ell] = \frac{\binom{b}{\ell}\binom{n-b}{k-\ell}}{\binom{n}{k}}.$$

Therefore,

$$\mathbb{E}[B] = \sum_{\ell=0}^{k} \frac{\ell\binom{b}{\ell}\binom{n-b}{k-\ell}}{\binom{n}{k}}.$$

One way to sum this series is the following: first, note that for $\ell \le b$, $\ell\binom{b}{\ell} = b\binom{b-1}{\ell-1}$ since

$$\ell\binom{b}{\ell} = \frac{\ell b!}{(b-\ell)!\ell!} = \frac{b(b-1)!}{(b-\ell)!(\ell-1)!} = b\binom{b-1}{\ell-1}.$$

Therefore,

$$\begin{aligned}
\mathbb{E}[B] &= \sum_{\ell=0}^{k} \frac{b\binom{b-1}{\ell-1}\binom{n-b}{k-\ell}}{\binom{n}{k}} \\
&= b\sum_{\ell=0}^{k} \frac{\binom{b-1}{\ell-1}\binom{n-b}{k-\ell}}{\frac{n}{k}\binom{n-1}{k-1}} \\
&= \frac{bk}{n}\sum_{\ell=0}^{k} \frac{\binom{b-1}{\ell-1}\binom{(n-1)-(b-1)}{(k-1)-(\ell-1)}}{\binom{n-1}{k-1}} \\
&= \frac{bk}{n}.
\end{aligned}$$

The last equality holds because $\sum_{\ell=0}^{k}\binom{b-1}{\ell-1}\binom{(n-1)-(b-1)}{(k-1)-(\ell-1)} = \binom{n-1}{k-1}$ – both sides count the number of ways to choose $k-1$ apples (without replacement) from a pile of $n-1$ apples, of which $b-1$ are bad. Note that this is the same answer as above!

Method 2: Use linearity of expectation.

Again, linearity of expectation is much simpler. Let X_i denote the random variable which is equal to 1 if the i^{th} apple we choose is bad, and is equal to 0 if the i^{th} apple we choose is good. Then, the number of bad apples we choose is equal to $X_1 + \cdots + X_k$ i.e. $B = X_1 + \cdots + X_k$ Hence, the expected number of bad apples is

$$\mathbb{E}[B] = \mathbb{E}[X_1 + \cdots + X_k] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_k].$$

We will show that it is still the case that $\mathbb{E}[X_i] = b/n$ for all i. In order to see this, note that there are $\binom{n}{k}k!$ sequences representing the k apples chosen (in order). We want to count the number of such sequences which contain a bad apple in position i. There are b ways to choose such a bad apple. The remaining $k-1$ positions can have any of the remaining $n-1$ apples, hence there are $\binom{n-1}{k-1}(k-1)!$ such sequences of length $k-1$. Put together, we see that

$$\begin{aligned}\mathbb{E}[X_i] &= \frac{b\binom{n-1}{k-1}(k-1)!}{\binom{n}{k}k!} \\ &= \frac{b(n-1)!/(n-k)!}{n!/\{(n-k)!\}} \\ &= \frac{b}{n}.\end{aligned}$$

Therefore,

$$\mathbb{E}[B] = \mathbb{E}[X_1 + \cdots + X_k] = \frac{kb}{n}.$$

Method 3: Another way of using linearity of expectation.

For $1 \le i \le b$, let Y_i be the random variable which is equal to 1 if the i^{th} bad apple is chosen and 0 otherwise. Then, the number of bad apples chosen is precisely $B = Y_1 + \cdots + Y_b$. Therefore, by the linearity of expectation

$$\mathbb{E}[B] = \mathbb{E}[Y_1 + \cdots + Y_b] = \mathbb{E}[Y_1] + \cdots + \mathbb{E}[Y_b].$$

Finally, note that for all $1 \le i \le b$, $\mathbb{E}[Y_i] = \mathbb{P}[Y_i = 1] = \frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{k}{n}$, since there are $\binom{n}{k}$ total ways of choosing k apples, and there are $\binom{n-1}{k-1}$ ways of choosing k apples such that the i^{th} bad apple is also chosen. Therefore, as before

$$\mathbb{E}[B] = \mathbb{E}[Y_1 + \cdots + Y_b] = \frac{bk}{n}.$$

Solution Optional Problem. Let T denote the number of tosses until the first tail appears. Let W denote your winnings (in dollars). Then, $W = 2^T$.

(i) We can easily compute the pmf of T; indeed, for any integer $i \ge 1$, we have

$$\mathbb{P}[T = i] = 2^{-i}.$$

Therefore,

$$\mathbb{E}[W] = \sum_{i=1}^{\infty} 2^i 2^{-i} = \sum_{i=1}^{\infty} 1 = \infty.$$

(ii) Recall that $F_W(x) = \mathbb{P}[W \le x]$. In our case, note that $F_W(x) = 0$ for all $x < 2$. For $x \ge 2$, let k denote the largest integer such that $2^k \le x$. Then, we have

$$F_W(x) = \mathbb{P}[W \le x] = \mathbb{P}[T \le k] = \sum_{j=1}^{k} \mathbb{P}[T = j] = \sum_{j=1}^{k} 2^{-j} = \left(1 - 2^{-k}\right).$$