18.600 Recitation 4

Recitation Instructor: Vishesh Jain
math.mit.edu/∼visheshj
Thursday, Oct. 4th, 2018

Solution Problem 1. Each bag can be thought of as a sequence of length 10 made up of the characters $\{W, M, D, C\}$. Since each such sequence is equally likely, both parts reduce to counting problems. Note that there are $4^{10} = 2^{20}$ such sequences.

(a) In this part, we need to count sequences with exactly 3 Ms. There are $\binom{10}{3}$ ways of choosing three positions for the Ms, and 3^7 choices for the characters in the remaining 7 positions. Hence, there are $\binom{10}{3} \times 3^7$ such sequences, and the desired probability is

$$\frac{\binom{10}{3} \times 3^7}{2^{20}} \approx 0.2502.$$

(b) We need to count the number of sequences which have either 0 or 1 Cs, or either 9 or 10 Cs. The number of sequences with 0 Cs is 3^{10} and the number of sequences with 10 Cs is 1. To count the number of sequence with 1 C, note that we can choose the position of the C in 10 ways, and then we have 3^9 ways of choosing the character in the remaining 9 positions; hence, there are 10×3^9 such sequences. Similarly, the number of sequences with 9 Cs is 10×3. Putting everything together, we see that the number of such sequences is

$$3^{10} + 1 + 10 \times 3^9 + 30 = 255910,$$

and the probability that a bag is defective equals

$$\frac{255910}{2^{20}} \approx 0.244.$$

Solution Problem 2. Note that X is a geometric random variable with parameter $p = 1/3$. Hence, $\mathrm{E}[X] = 3$. To compute $\mathrm{P}[X > 5]$, we can use two methods. The first method is to notice that the event $X > 5$ is precisely the event that Alice wins each of the first 5 games – this happens with probability $(2/3)^5$. The second method is

$$\begin{aligned}
\mathrm{P}[X > 5] &= \sum_{k=6}^{\infty} \mathrm{P}[X = k] \\
&= \sum_{k=6}^{\infty} \left(\frac{2}{3}\right)^{k-1} \left(\frac{1}{3}\right) \\
&= \left(\frac{1}{3}\right) \sum_{\ell=5}^{\infty} \left(\frac{2}{3}\right)^{\ell} \\
&= \left(\frac{2}{3}\right)^{5}.
\end{aligned}$$

Solution Problem 3. Note that X is a geometric random variable with parameter $1/6$ and Y is distributed as $\mathrm{Bin}(4, 1/3)$.

(a) By linearity of expectation, we know that $\mathrm{E}[Z] = \mathrm{E}[X] + \mathrm{E}[Y]$. Moreover, $\mathrm{E}[X] = 6$ and $\mathrm{E}[Y] = 4/3$. Therefore, $\mathrm{E}[Z] = 22/3$.

(b) $Z = 2$ can happen only via the following disjoint cases: $X = 2$ and $Y = 0$ or $X = 1$ and $Y = 1$. By the independence of X and Y, the probability that $X = 2$ and $Y = 0$ is

$$\left(\frac{5}{6} \times \frac{1}{6}\right) \times \left(\frac{2}{3}\right)^4 = \frac{20}{3^6}.$$

Similarly, the probability that $X = 1$ and $Y = 1$ is

$$\left(\frac{1}{6}\right) \times \left(4 \times \left(\frac{2}{3}\right)^3 \times \frac{1}{3}\right) = \frac{48}{3^6}.$$

Hence, we get that

$$\mathrm{P}[Z = 2] = \frac{68}{3^6}.$$

(c) Note that $F_Z(2) = \mathrm{P}[Z \leq 2] = \mathrm{P}[Z = 1] + \mathrm{P}[Z = 2]$, since we know that Z is an integer, and Z cannot be less than 1 (as $X \geq 1$ by definition). We have already computed $\mathrm{P}[Z = 2]$, so it only remains to compute $\mathrm{P}[Z = 1]$. This can happen only if $X = 1$ and $Y = 0$, which happens with probability

$$\left(\frac{1}{6}\right) \times \left(\frac{2}{3}\right)^4 = \frac{24}{3^6}.$$

Hence, we see that

$$F_Z(2) = \frac{92}{3^6}.$$

Solution Problem 4. **(a)** The probability that both teams get a question correct is $3/4 \times 2/3 = 1/2$. Therefore, X is distributed as $\mathrm{Bin}(10, 1/2)$, and hence, $E[X] = 5$.

(b) By inclusion-exclusion (in this case, $\mathrm{P}[A \cup B] = \mathrm{P}[A] + \mathrm{P}[B] - \mathrm{P}[A \cap B]$), the probability that at least one team gets a question correct is $3/4 + 2/3 - 1/2 = 11/12$. Therefore, Y is distributed as $\mathrm{Bin}(10, 11/12)$, and hence, $\mathrm{Var}[Y] = 10 \times \frac{11}{12} \times \left(1 - \frac{11}{12}\right) = 110/144$.

(c) We know that $\mathrm{E}[Y^2] = \mathrm{Var}[Y] + (\mathrm{E}[Y])^2$. From the previous part, we know that $\mathrm{Var}[Y] = 110/144$. Moreover, since Y is distributed as $\mathrm{Bin}(10, 11/12)$, we know that $\mathrm{E}[Y] = 110/12$. Hence, we get that $\mathrm{E}[Y^2] = 110/144 + (110/12)^2 = (110^2 + 110)/144 = (110 \times 111)/144$.