# 18.600 Recitation 5

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**Solution Problem 1.** **(a)** There are $\binom{r+b}{3}$ ways to choose 3 balls from the urn. There are $\binom{r}{3}$ ways to choose 3 red balls, and $\binom{r}{2}\binom{b}{1}$ ways to choose 2 red balls and 1 blue ball. Therefore, the required probability is

$$\frac{\binom{r}{3} + \binom{r}{2}\binom{b}{1}}{\binom{r+b}{3}}.$$

**(b)** Think about numbering the blue balls as well (actually, this is what was implicitly done in the last part). There are $(r+b)!$ ways of arranging the balls in a sequence. Let us count the number of such sequences in which $R_i$ appears before any $B_j$. There are $\binom{r+b}{b+1}$ ways of choosing the $b+1$ positions for $B_1, \ldots, B_b, R_i$, followed by $b!$ ways of arranging these balls (note that $R_i$ must appear in the first position, and there are $b!$ ways of arranging the blue balls), followed by $(r-1)!$ ways of arranging the remaining red balls. Hence, the required probability is

$$\frac{\binom{r+b}{b+1} b!(r-1)!}{(r+b)!} = \frac{1}{b+1}.$$

Note that we could also have obtained this answer directly by symmetry considerations – any of the $b+1$ balls $R_i, B_1, \ldots, B_b$ is equally likely to be in the first position.
**(c)** Let $X_i$ denote the indicator variable for the event $E_i$. The number of red balls drawn before the first blue ball equals $X := X_1 + \cdots + X_r$. Therefore, by the linearity of expectation, $\mathrm{E}[X] = \mathrm{E}[X_1] + \cdots + \mathrm{E}[X_r] = r/(b+1)$.

**Solution Problem 2.** **(a)**. Note that $X \sim \mathrm{Bin}(1000, 0.998)$. Therefore, $\mathrm{E}[X] = 998$ and $\mathrm{Var}[X] = 1000 \times .998 \times 0.002 = 1.996$.
**(b)** The system works if 0 components fail or 1 component fails or 2 components fail. Adding the probabilities of these events, we see the required probability is

$$\mathrm{P}[W] = (0.998)^{1000} + \binom{1000}{1}(0.998)^{999}(0.002) + \binom{1000}{2}(0.998)^{998}(0.002)^2.$$

**(c)** Let $W$ be the event that the system works, and let $W_1$ be the event that component 1 works. Then, we have

$$\begin{aligned} \mathrm{P}[W_1|W] &= \mathrm{P}[W_1 \cap W]/\,\mathrm{P}[W] \\ &= \mathrm{P}[W|W_1]\,\mathrm{P}[W_1]/\,\mathrm{P}[W]. \end{aligned}$$

We computed $\mathrm{P}[W]$ in the previous part, and we know that $\mathrm{P}[W_1] = 0.998$. So, the only term that remains to be computed is $\mathrm{P}[W|W_1]$. By a similar computation as in the previous part, we have

$$\mathrm{P}[W|W_1] = (0.998)^{999} + \binom{999}{1}(0.998)^{998}(0.002) + \binom{999}{2}(0.998)^{997}(0.002)^2.$$

**(d)** The probability that the first 500 components all work is $(0.998)^{500}$. The number of components which fail in the second five hundred components is distributed as $\mathrm{Bin}(500, 0.002)$. We can approximate this by the Poisson distribution with parameter $\lambda = 500 \times 0.002 = 1$.

The probability of this distribution being exactly two is $e^{-1}/2$. Therefore, the required probability is approximately $(0.998)^{500}/(2e)$.

**Solution Problem 3.** **(a)** The sum of the three dice is odd if and only if either every dice is odd, or two of them are even and the third is odd. Therefore, we have

$$E = RGB \cup R(G^c)(B^c) \cup R(G^c)(B^c) \cup (R^c)(G^c)B.$$

**(b)** Note that $X$ is distributed as $\mathrm{Bin}(3, 1/2)$, and we have computed the cumulative distribution function of binomial random variables several times.
**(c)** Since $X \sim \mathrm{Bin}(3, 1/2)$, $\mathrm{E}[X] = 3/2$.

**Solution Problem 4.** **(a)** Note that $X = 15 + 2Y$, where $Y \sim \mathrm{Bin}(10, 0.25)$ We can compute the pmf of $X$ directly using this characterization.
**(b)** By linearity of expectation, $\mathrm{E}[X] = \mathrm{E}[15 + 2Y] = 15 + 2\,\mathrm{E}[Y] = 15 + 5 = 20$. Also, $\mathrm{Var}[X] = \mathrm{Var}[15 + 2Y] = \mathrm{Var}[2Y] = 4\mathrm{Var}[Y] = 4 \times 10 \times 0.25 \times 0.75$.
**(c)** When Alice walks, we know that the pmf is 1 at 20, and 0 everywhere else. We computed the pmf when she drives in part (a). The pmf of $Y$ is obtained by taking a weighted average of these two pmfs, with weights 2/3 and 1/3 respectively.