

18.600 Recitation 6

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Solution Problem 1. (a) He gets to Smallville in less than 1.5 hours if and only if his speed is at least $1000/(1.5) = 2000/3$ mph. The speed is distributed as $\max\{0, X\}$ where $X \sim N(500, 100^2)$. Note that X has the same distribution as $500 + 100Y$, where $Y \sim N(0, 1)$. Therefore,

$$\Pr[S > 2000/3] = \Pr[500 + 100Y > 2000/3] = \Pr[Y > 5/3] = 1 - \Pr[Y \leq 5/3] = 1 - \Phi(5/3).$$

(b) First, we compute the CDF of T . By the same argument as above, we know that for any $t > 0$,

$$\Pr[T \leq t] = \Pr[S \geq 1000/t] = 1 - \Phi((10/t) - 5).$$

Therefore, for any finite $t > 0$,

$$p_T(t) = \frac{10}{t^2} \Phi'((10/t) - 5) = \frac{10 \exp(-((10/t) - 5)^2/2)}{t^2 \sqrt{2\pi}}.$$

Finally, note that T has probability mass $\Phi(-5)$ at $t = \infty$.

Solution Problem 2. The probability that the first 500 components all work is $(0.998)^{500}$. The number of components which fail in the second five hundred components is distributed as $\text{Bin}(500, 0.002)$. We can approximate this by the Poisson distribution with parameter $\lambda = 500 \times 0.002 = 1$. The probability of this distribution being exactly two is $e^{-1}/2$. Therefore, the required probability is approximately $(0.998)^{500}/(2e)$.

Solution Problem 3. (a) The number of earthquakes occurring in the next two weeks is distributed as a Poisson random variable with parameter 4. Therefore, the probability that there are at least 3 earthquakes in the next 2 weeks is

$$1 - e^{-4} - 4e^{-4} - 8e^{-4} = 1 - 13e^{-4}.$$

(b) Let T be the number of weeks until the next earthquake. Then,

$$\Pr[T > t] = \Pr[\text{No earthquakes occur in the next } t \text{ weeks}] = e^{-2t}.$$

Therefore,

$$F_T(t) = \Pr[T \leq t] = 1 - \Pr[T > t] = 1 - e^{-2t}.$$

Solution Problem 4. (a) $E[Y] = \int_0^2 e^{x^2} \left(\frac{x}{2}\right) dx = \frac{1}{4}(e^4 - 1)$.

(b) $E[Y^2] = \int_0^2 e^{2x^2} \left(\frac{x}{2}\right) dx = \frac{1}{8}(e^8 - 1)$. Therefore,

$$\text{Var}[Y] = E[Y^2] - (E[Y])^2 = \frac{e^8}{16} + \frac{e^4}{8} - \frac{3}{16}.$$

(c) Note that for any $t < 1$, $F_Y(t) = 0$. Also, for $t \geq 1$,

$$F_Y(t) = \Pr[Y \leq t] = \Pr[X \leq \sqrt{\log t}] = F_X(\sqrt{\log t}).$$

Hence, $F_Y(t) = 1$ for $t \geq e^4$, whereas for $1 \leq t \leq e^4$, we have

$$F_Y(t) = \int_0^{\sqrt{\log t}} \frac{x}{2} dx = \frac{\log t}{4}.$$

(d) By differentiating the CDF obtained above, we see that $p_Y(t) = 0$ if $t < 1$ or if $t > e^4$. For $1 \leq t \leq e^4$, we have $p_Y(t) = \frac{1}{4t}$.

Solution Problem 5. Let I denote the IQ of a random person. We are given that $I \sim N(100, 225)$. In terms of the standard normal distribution, we can write $I \sim 100 + 15Y$, where $Y \sim N(0, 1)$. Our goal is to find x such that $\Pr[I \leq x] = 0.98$. Rewriting this in terms of Y , we have $\Pr[100 + 15Y \leq x] = 0.98$ i.e. $\Pr[Y \leq (x - 100)/15] = 0.98$. Hence, the desired IQ is the value of x satisfying $\Phi((x - 100)/15) = 0.98$.

Solution Problem 6. The three segments can form a triangle if and only if the sum of lengths of any two of the sides is at least the length of the third side. Here's one way of doing this problem (which is not the cleanest). Call the points A and B . There are two cases, each of which contributes equally: when point A lands in the interval $(0, 1/2)$, and when point A lands in the interval $(1/2, 1)$. In the first case, if point A lands at x , then point B must land somewhere in the interval $1/2, 1/2 + x$, which happens with probability x . Therefore, in this case, the probability that the sides form a triangle is

$$\int_0^{1/2} x dx = \frac{1}{8}.$$

The other case similarly contributes $1/8$ to the probability. Therefore, the probability that the three lengths form a triangle is $1/4$.