18.600 Recitation 7

Recitation Instructor: Vishesh Jain

Partial solutions available at math.mit.edu/∼visheshj

Thursday, Oct. 25th, 2018

Solution Problem 1. **(a)** The joint density function is $f(11{:}\,00+a, 11{:}\,00+b) = 1/1800$ for $15 < a < 45, 0 < b < 60$, and 0 for other values of a, b. Notice that the person who arrives first waits for less than 5 minutes if and only if Bob arrives within 5 minutes of Alice. The probability of this is given by

$$\int_{15}^{45}\int_{a-5}^{a+5}(1/1800)dbda = 1/6.$$

(b) If Bob arrives between 11:00 and 11:15 (which happens with probability 1/4), then he is always first, and if he arrives between 11:15 and 11:45 (which happens with probability 1/2), then by symmetry, he arrives first with probability 1/2. Therefore, the total probability that Bob arrives first is $\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{2}$.

Solution Problem 2. **(a)** Let A denote the number of days it takes for Alice to get her car serviced, and let B denote the number of days it takes for Bob to get his car serviced. Then,

$$\Pr[B > A] = \int_0^\infty\int_a^\infty \frac{1}{10}\exp(-b/2)\exp(-a/5)dbda = \int_0^\infty \frac{1}{5}\exp(-7a/10) = 2/7.$$

(b) Since the exponential distribution is memoryless, conditioned on $B > 10$, B is still exponentially distributed. Therefore, the expected number of further days Bob has to wait until his car is serviced is 2.

(c) Note that $T = A + B$. Therefore, for any $t > 0$, we have

$$f_T(t) = \int_0^t \frac{1}{10}\exp(-x/2)\exp(-(t-x)/5)dx = \frac{1}{3}\left(e^{-t/5}-e^{-t/2}\right).$$

Solution Problem 3. **(a)** Since

$$\int_0^1\int_0^2 \left(x^2+(xy)/2\right)dydx = 7/6,$$

it follows that a must be 6/7.

(b) $f_X(x) = 0$ for $x \notin (0,1)$. For $x \in (0,1)$, we have

$$f_X(x) = \int_0^2 a\left(x^2+(xy)/2\right)dy = a\left(2x^2+x\right).$$

(c) We have

$$E[X] = \int_0^1 x\cdot a(2x^2+x)dx = 5/7.$$

(d) No, because no function of y multiplied by $f_X(x)$ can give the joint density given in the problem.

(e) $\int_0^{1/2}\int_{1/2}^2 a\,(x^2+(xy)/2)\,dydx = 69/448.$
(f) We need to divide the answer above by

$$P[X<1/2] = \int_0^{1/2} a(2x^2+x)dx = 5/28.$$

This shows that the desired conditional probability is 69/80.

Solution Problem 4. Note that $f_{X+Y}(a)=0$ if $a\le 0$. For $a>0$, we have

$$\begin{aligned} f_{X+Y}(a) &= \int_0^2 f_{X,Y}(x,a-x)dx \\ &= \frac{1}{2}\int_0^2 \mathbf{1}_{0\le a-x\le 2-x}dx \\ &= \frac{1}{2}\int_0^2 \mathbf{1}_{a\le 2}\cdot \mathbf{1}_{x\le a}dx \\ &= \frac{1}{2}\int_0^a \mathbf{1}_{a\le 2}dx \\ &= \begin{cases} \frac{a}{2} & 0\le a\le 2; \\ 0 & \text{else.} \end{cases} \end{aligned}$$