18.600 Recitation 8

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Partial solutions available at math.mit.edu/∼visheshj

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Solution Problem 1. **(a)** The area of the triangle is 1/2. Hence, the joint PDF is 2 inside the triangle, and 0 outside the triangle.

(b) The marginal PDF is 0 unless $y \in (0,1)$. For $y \in (0,1)$, we have $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \int_0^{1-y} 2dx = 2(1-y)$.

(c) $f_{X,Y}(x,y) = f_{X|Y}(x|y)f_Y(y)$. Therefore, we see from the first two parts that $f_{X|Y}(x|y) = 1/(1-y)$ for (x,y) inside the triangle. This is also clear geometrically.

(d) $E[X|Y=y] = \int_0^{1-y} \frac{x}{(1-y)}dx = \frac{1-y}{2}$. Therefore, $E[X] = \int_0^1 E[X|Y=y]f_Y(y)dy = \frac{1}{2} - \frac{E[Y]}{2}$.

(e) By symmetry, $E[X] = E[Y]$. So, the previous part gives $3E[X] = 1$ i.e. $E[X] = \frac{1}{3}$.

Solution Problem 2. **(a)** For $1 \leq i \leq 13$, let X_i denote the indicator for the i^{th} card being an ace, and let Y_i denote the indicator for the i^{th} card being a spade. Then, $X = X_1 + \cdots + X_{13}$ and $Y = Y_1 + \cdots + Y_{13}$. Since $\text{Cov}(X,Y) = \sum_{i,j}\text{Cov}(X_i,Y_j)$, it is sufficient to show that $\text{Cov}(X_i,Y_j) = 0$ for all pairs (i,j), and this is easily checked directly using the definition of covariance.

(b) X and Y are not independent, since $0 = \Pr[X=2, Y=13] \neq \Pr[X=2]\Pr[Y=13]$.

Solution Problem 3. We use the trick that $N|N>k \sim N+k$. Specifically, we will use it for $k=2$. Let $X = \cos(N\pi)$. Then, we have

$$\begin{aligned}
E[X] &= E[X|N=1]\Pr[N=1] + E[X|N=2]\Pr[N=2] + E[X|N>2]\Pr[N>2] \\
&= \cos(\pi)p + \cos(2\pi)(1-p)p + (1-p)^2E[X|N>2] \\
&= \cos(\pi)p + \cos(2\pi)(1-p)p + (1-p)^2E[\cos((N+2)\pi)] \\
&= \cos(\pi)p + \cos(2\pi)(1-p)p + (1-p)^2E[\cos(N\pi)] \\
&= \cos(\pi)p + \cos(2\pi)(1-p)p + (1-p)^2E[X].
\end{aligned}$$

Solving for $E[X]$ gives $E[X] = p/(p-2)$.

Solution Problem 4. The sample space here is sequences of dice rolls. Let's compute the probability of the event that all throws until the first 6 are even. Using the law of total probability, this is

$$\sum_{k=1}^{\infty}\left(\frac{2}{5}\right)^{k-1}\left(\frac{5}{6}\right)^{k-1}\left(\frac{1}{6}\right) = \frac{1}{4}.$$

This calculation also shows that the the probability of the sequences where the first 6 appears in the k^{th} position, and all preceding entries are even is $\left(\frac{2}{6}\right)^{k-1}\frac{1}{6}$. Therefore, the expected number of trials until the first 6, conditioned on this event, is

$$\sum_{k=1}^{\infty} k\left(\frac{1}{3}\right)^{k-1}\frac{(1/6)}{(1/4)} = \sum_{k=1}^{\infty} k\left(\frac{1}{3}\right)^{k-1}\left(\frac{2}{3}\right) = \frac{3}{2}.$$

The probability that the first die is 6, conditioned on the event, is $(1/6)/(1/4) = 2/3$. This is consistent with the previous answer, since it shows that the conditional distribution amounts to rolling a 3 sided die (with faces 2, 4, 6), where the probability of rolling a 6 is 2/3, and the goal is to find the expected number of rolls until the first 6.