18.600 Recitation 9

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Partial solutions available at math.mit.edu/∼visheshj

Thursday, Nov. 8th, 2018

Solution Problem 1. Spring 2013, 2(d). Note that the solution there shows that the answer to both questions is no.

Solution Problem 2. Fall 2015, 1(c).

Solution Problem 3. **(a)** $f_{X,Y}(a,b) = 1/\pi$ if $a^2 + b^2 \leq 1$, and 0 otherwise.
(b),(c),(d) 0, by symmetry arguments.

Solution Problem 4. Based on Spring 2015, 4(a),(b),(c).
(a) Since X and Y are independent, $3X - 6Y$ is a normal variable with mean -27, and variance 225, and we already know the PDF of normal random variables with arbitrary mean and variance.
(b) We start by computing the CDF of $W = |2X+1|$. For any $t < 0$, $\Pr[W \leq t] = 0$, and for any $t > 0$, we have $\Pr[W \leq t] = \Pr[-t \leq (2X+1) \leq t] = \Pr[-(t+1) \leq 2X \leq (t-1)]$. Since $2X \sim 2 + 6N$, where N is distributed as the standard normal variable, it follows that this probability equals $\Pr[-(t+3)/6 \leq N \leq (t-3)/6] = \Phi((t-3)/6) - \Phi(-(t+3)/6) = \Phi((t-3)/6) + \Phi((t+3)/6) - 1$. Now, differentiate this to get the PDF.
(c) Again, we start by computing the CDF of $Z = \max\{X, Y\}$. Since X and Y are independent, we have $\Pr[Z \leq t] = \Pr[X \leq t \text{ and } Y \leq t] = \Pr[X \leq t] \cdot \Pr[Y \leq t] = \Phi((t-1)/3) \cdot \Phi((t-5)/2)$. Now, differentiate this to get the PDF.

Solution Problem 5. Based on Fall 2015, 4.
(a) No. One way to see this is to note that $f_{Y|X}(b|a)$ depends on a.
(b) 0, since $E[Y] = E[E[Y|X]]$ and note that $E[Y|X=a] = 0$ for all $a \in [0,1]$.
(c) $\mathrm{Var}[Y] = E[Y^2] - (E[Y])^2 = E[Y^2]$, using the previous part. Also, $E[Y^2] = E[E[Y^2|X]]$. Note that $E[Y^2|X=a] = \frac{1}{2a}\int_{-a}^{a} b^2 db = 2a^3/6a = a^2/3$. Therefore, $\mathrm{Var}[Y] = E[Y^2] = \int_0^1 a^2/3 = 1/9$.

Solution Problem 6. **(a)** A simple example is $X \sim \mathrm{Ber}(1/2)$, $Y \sim \mathrm{Unif}[-1,1]$, X and Y independent, and $Z = XY$. Then, X and Z are uncorrelated but not independent (why?).

(b) Yes. Since $E[XY] = \Pr[X=1, Y=1]$, $E[X] = \Pr[X=1]$ and $E[Y] = \Pr[Y=1]$, we see that $\Pr[X=1, Y=1] = \Pr[X=1] \cdot \Pr[Y=1]$. To see that $\Pr[X=1, Y=0] = \Pr[X=1] \cdot \Pr[Y=0]$, note that $\Pr[X=1, Y=0] = E[X(1-Y)] = E[X - XY] = E[X] - E[X] \cdot E[Y] = E[X](1 - E[Y]) = \Pr[X=1]\Pr[Y=0]$. There are two more cases, which can be checked similarly.

(c) It is equal to $ac\mathrm{Cov}(X,Y)$, as can be directly checked from definition, or from the bilinearity of covariance.