Let $X_1, \dots, X_n$ be non-negative independent random variables, each of which has mean 1. Let $X = X_1 + \dots + X_n$. We will show (following Nazarov https://mathoverflow.net/questions/187938/lower-bound-for-prx-geq-ex/188087#188087) that there exists an absolute constant $c > 0$ such that

$$\Pr\left(X \le n + 1\right) \ge c.$$

We will find it more convenient to work instead with $Y_i = 1 - X_i$. Note that $Y_i \le 1$ and $\mathbb{E}[Y_i] = 0$ for all i. The above statement is equivalent to showing that

$$P := \Pr\left(Y \ge -1\right) \ge c.$$

Proof: The proof will proceed by case analysis, depending on the value of $\mathbb{E}\, e^Y$.

Case I: $\mathbb{E}\, e^Y \le 2$. In this case, we have

$$1 \le e^{\mathbb{E}[Y]/2} \le \mathbb{E}\, e^{Y/2} = \mathbb{E}[e^{Y/2}\mathbf{1}(Y < -1)] + \mathbb{E}[e^{Y/2}\mathbf{1}(Y \ge -1)] \le e^{-1/2}(1 - P) + \sqrt{2P};$$

the second inequality is Jensen's inequality, and the rightmost inequality is by Cauchy-Schwarz. Hence, in this case, we get that

$$1 \le e^{-1/2}(1 - P) + \sqrt{2P}$$

which translates to $P \gtrapprox 0.104$.

Case II: $\mathbb{E}\, e^Y > 2$. In this case, there exists some $t \in (0, 1)$ such that $\mathbb{E}\, e^{tY} = 2$. We claim that there exists some absolute constant $K > 1$ for which $\mathbb{E}\, e^{2tY} \le 2^K$. Before proving this claim, let's see how this finishes the proof.

Setting $q := 2^{-K-1}$, we have

$$\mathbb{E}\left[e^{tY} - qe^{2tY} - 1\right] \ge 2 - 2^{-1} - 1 \ge \frac{1}{2}.$$

Moreover, the function $x - qx^2 - 1$ is bounded above by $1/4q = 2^{K-1}$, and is negative whenever $x < 0$. Therefore, we have

$$\frac{1}{2} \le \mathbb{E}\left[e^{tY} - qe^{2tY} - 1\right] \le 2^{K-1}\Pr(Y \ge 0) \le 2^{K-1}P.$$

It remains to prove the claim. It suffices to show that there exists some absolute constant $K > 1$ such that if $Z \le 1$ is a mean zero random variable, then

$$\mathbb{E}\, e^{2Z} \le (\mathbb{E}\, e^Z)^K.$$

From this, the claim follows since

$$\begin{aligned}
\mathbb{E}\, e^{2tY} &= \prod_{i=1}^n \mathbb{E}\, e^{2tY_i} \\
&\le \left(\prod_{i=1}^n \mathbb{E}\, e^{tY_i}\right)^K \\
&= \left(\mathbb{E}\, e^{tY}\right)^K = 2^K.
\end{aligned}$$

Finally, the inequality for Z follows by noting that there is some absolute constant $K > 1$ for which the following numerical inequality is true: $e^{2z} - 1 - 2z \leq K\,(e^z - 1 - z)$ for all $z \leq 1$, and the following chain of inequalities:

$$\begin{aligned}\left(\mathbb{E}\, e^Z\right)^K &= \left(1 + \mathbb{E}[e^Z - 1 - Z]\right)^K \\ &\geq 1 + K\,\mathbb{E}[e^Z - 1 - Z] \\ &\geq 1 + \mathbb{E}[e^{2Z} - 1 - 2Z] \\ &= \mathbb{E}[e^{2Z}].\end{aligned}$$

Application: A *fractional matching* in a k-graph $H = (V, E)$ is a function $w : E \to [0, 1]$ such that for every $v \in V$, $\sum_{e \ni v} w(e) \leq 1$ (observe that if $w : E \to \{0, 1\}$, then the same condition gives a matching). The *size* of a fractional matching is defined to be $\sum_{e \in E} w(e)$. We say that w is a *perfect fractional matching* if its size is $|V|/k$ (or equivalently, if $\sum_{e \ni v} w(e) = 1$ for all $v \in V$).

For an integer $0 \leq d \leq k-1$ and a real number $0 \leq s \leq n/k$, we let $f_d^s(k, n)$ denote the smallest integer m such that every n-vertex k-graph H with $\delta_d(H) \geq m$ has a fractional matching of size s. We denote $f_d^{n/k}(k, n)$ simply by $f_d(k, n)$. Also, let

$$f_d(k) := \limsup_{n \to \infty} \frac{f_d(k, n)}{\binom{n-d}{k-d}}.$$

It was proved by Alon, Frankl, Huang, Rödl, Ruciński, and Sudakov that for all $k \geq 3$ and $1 \leq d \leq k-1$,

$$f_d(k) \leq f^d(k - d),$$

where

$$f^d(\ell) := \limsup_{m \to \infty} \frac{f_0^{m+d/\ell+d}(\ell, m)}{\binom{m}{\ell}}.$$

Recently, together with Asaf Ferber, we observed that

$$f^d(\ell) = \Theta^d(\ell).$$

Here

$$\Theta^d(\ell) := \sup \Pr[X_1 + \cdots + X_\ell \geq \ell + d],$$

where the supremum is taken over all collections of non-negative i.i.d. random variables $X_1, \ldots, X_\ell$ with mean 1.