April 20, 2018.

Hypergraph containers theorem I: Let $k \in \mathbb{N}$ be fixed. For all $c, \epsilon > 0$, there exists $C > 0$ for which the following holds. Let $\mathcal{H}$ be a k-uniform hypergraph, and let $\mathcal{F}$ be an increasing family of sets such that $|A| \geq \epsilon v(\mathcal{H})$ for all $A \in \mathcal{F}$. Suppose $\mathcal{H}$ is $(\mathcal{F}, \epsilon)$ dense (i.e. $e(\mathcal{H}[A]) \geq \epsilon e(\mathcal{H})$ for all $A \in \mathcal{F}$), and $p \in (0,1)$ is such that for all $\ell \in [k]$,

$$\Delta_\ell(\mathcal{H}) \leq c.p^{\ell-1}.\frac{e(\mathcal{H})}{v(\mathcal{H})}.$$

Then, there exists a family $\mathcal{S} \subseteq \binom{V(\mathcal{H})}{C.pv(\mathcal{H})}$ and functions $f\colon \mathcal{S} \to \overline{\mathcal{F}}$ and $g\colon \mathcal{I}(\mathcal{H}) \to \mathcal{S}$ such that for all $I \in \mathcal{I}(\mathcal{H})$, $g(I) \subseteq I$ and $I \backslash g(I) \subseteq f(g(I))$.

The canonical example of increasing family $\mathcal{F}$ to keep in mind is the set of all subsets of $V(\mathcal{H})$ of size $\geq (1-\delta)v(\mathcal{H})$ for some fixed $\delta > 0$. In this case, the hypergraph containers theorem says that if $\mathcal{H}$ is a k-uniform hypergraph satisfying certain technical assumptions, then each independent set I of $\mathcal{H}$ can be fingerprinted using a small subset S such that knowledge of the fingerprint S identifies a positive constant fraction of the vertices that I avoids. In fact, the theorem above for general increasing families follows from the next theorem for this particular choice of increasing family.

Hypergraph containers theorem II: Let $k \in \mathbb{N}$ be fixed. For all $c_0 > 0$, there exists $\delta > 0$ for which the following holds. Let $\mathcal{H}$ be a k-uniform hypergraph, and let $\mathcal{F}_0$ be the increasing family of sets such that $|A| \geq (1-\delta)v(\mathcal{H})$ for all $A \in \mathcal{F}_0$. Suppose $p_0 \in (0,1)$ is such that for all $\ell \in [k]$,

$$\Delta_\ell(\mathcal{H}) \leq c_0 p_0^{\ell-1}.\frac{e(\mathcal{H})}{v(\mathcal{H})}.$$

Then, there exists a family $\mathcal{S} \subseteq \binom{V(\mathcal{H})}{(k-1).pv(\mathcal{H})}$ and functions $f_0\colon \mathcal{S} \to \overline{\mathcal{F}_0}$ and $g_0\colon \mathcal{I}(\mathcal{H}) \to \mathcal{S}$ such that for all $I \in \mathcal{I}(\mathcal{H})$, $g_0(I) \subseteq I$ and $I \backslash g_0(I) \subseteq f_0(g_0(I))$. *Moreover, if for some $I, I' \in \mathcal{I}(\mathcal{H})$, $g_0(I) \subseteq I'$ and $g_0(I') \subseteq I$, then $g_0(I) = g_0(I')$.*

Let's quickly sketch how to deduce HCT I given HCT II. Let $c, \epsilon, \mathcal{F}, \mathcal{H}$ be as in the statement of HCT I. We describe an algorithm to construct $\mathcal{S}, f$ and g as in the conclusion of the theorem.

Algorithm (Input: $I \in \mathcal{I}(\mathcal{H})$): Let $S_0 = \emptyset$ and $A_0 = V$. For $j = 0, 1, \dots, J$ do the following:

1. If $A_j \in \mathcal{F}$, then apply HCT II to $\mathcal{H}[A_j]$ with the same p and with $c_0 = c/\epsilon$ to the set $I_j := I \cap A_j$ to get $g_0(I_j)$ and $f_0(g_0(I_j))$.

Why can we do this? Need to check that the assumptions of HCT II hold for the choice of parameters above. This follows since

$$\Delta_\ell(\mathcal{H}[A_j]) \le \Delta_\ell(\mathcal{H}[A]) \le c.p^{\ell-1}.\frac{e(\mathcal{H})}{v(\mathcal{H})} \le c.p^{\ell-1}.\frac{e(\mathcal{H})}{v(\mathcal{H}[A_j])} \le \frac{c}{\epsilon}.p^{\ell-1}\frac{e(\mathcal{H}[A_j])}{v(\mathcal{H}[A_j])}$$

where the last inequality uses the fact that $A_j \in \mathcal{F}$ and $\mathcal{H}$ is $(\mathcal{F}, \epsilon)$-dense.

2. Let $S_{j+1} = g_0(I_j)$ and let $A_{j+1} = f_0(g_0(I_j))$.

Observations:

(i) By construction, we have $S_0 \cup \dots \cup S_j \subseteq I \subseteq S_0 \cup \dots \cup S_j \cup A_j$ for all j.

(ii) $|A_{j+1}| \le (1-\delta)|A_j|$ for all j. Since any subset of $\mathcal{F}$ has size at least $\epsilon v(\mathcal{H})$ by assumption, it follows that we must terminate after at most $\frac{1}{\delta}\log\frac{1}{\epsilon}$ steps. So, we can take J to be this quantity.

(iii) The size of the fingerprint is $\le \sum_{i=1}^{J} |S_i| \le \sum_{i=0}^{J-1}(k-1)p|A_i| \le J(k-1)pv(\mathcal{H})$. Hence, we can take $C = J(k-1)$.

So, the only thing that remains to be checked is the following: if two independent sets I and I' have the same fingerprints $S_0 \cup \dots \cup S_J$ and $S_0' \cup \dots \cup S_J'$, then $A_J = A_J'$. We will show something even stronger. Let $(A_j, S_j)_{j=1}^{J}$ and $(A_j', S_j')_{j=1}^{J'}$ be the sequences generated by running the algorithm on I and I', and suppose $S_0 \cup \dots \cup S_J \subseteq I'$ and $S_0' \cup \dots \cup S_J' \subseteq I$. It is easy to see that this inductively implies $S_{j+1} \subseteq I_j'$ and $S_{j+1}' \subseteq I_j$ for all j. Therefore, $S_{j+1} = S_{j+1}'$ for all j by the conclusion of HCT II, and hence $A_J = A_J'$ as desired. So, we see that the "consistency" property of HCT II allows us to essentially decompose a complete fingerprint S into the smaller sets S_i from which it is algorithmically built.

Now, we move on to the proof of HCT II. As mentioned earlier, what we want to do is to fingerprint independent sets in such a way that the knowledge of the fingerprint helps us identify $\approx \delta$ fraction of the vertices that the independent set avoids. Intuitively, the idea is that the fingerprint S will consist of the vertices with large $k-1$ degree, so that once we know S, we know a large number of vertices which are forbidden from being in I. This motivates the so-called Scythe algorithm.

Scythe algorithm (Input: $\mathcal{H}_{i+1}$, $I \in \mathcal{I}(\mathcal{H}_{i+1})$). Let $\mathcal{A}_{i+1}^{(0)} = \mathcal{H}_{i+1}$ be the initial set of "available edges". Let $\mathcal{H}_i^{(0)}$ be the empty i-uniform hypergraph on the vertex set $V(\mathcal{H}_{i+1})$; this is the initial set of "forbidden edges". For $j = 0, 1, \ldots, b-1$, where $b = pv(\mathcal{H})$, do the following:

1. Let u_j be the first vertex of I in the max degree order on $V(\mathcal{A}_{i+1}^{(j)})$. If $I \cap V(\mathcal{A}_{i+1}^{(j)}) = \emptyset$ (think of this as "atypical" for the algorithm), then STOP with $\mathcal{H}_i = \mathcal{H}_i^{(0)}$, $A_i = \emptyset, B_i = \{u_0, \ldots, u_{j-1}\}$.

2. Add the edges incident to u_j to $\mathcal{H}_i^{(j)}$ in order to get $\mathcal{H}_i^{(j+1)}$.

3. Let $\mathcal{A}_{i+1}^{(j+1)}$ be the induced hypergraph on the vertex set $V(\mathcal{A}_{i+1}^{(j)}) \backslash W(u_j)$, where $W(u_j)$ is the set of all vertices which preceded u_j in the max degree order (so we know that these vertices are not in I) along with u_j.

4. Remove from $\mathcal{A}_{i+1}^{(j+1)}$ all edges which contain a set of vertices with "high degree" in $\mathcal{H}_i^{(j+1)}$ (this will be made precise at some point later on).

If we have not already stopped i.e. if the above loop executes b times, then set $\mathcal{H}_i = \mathcal{H}_i^{(b)}$, $A_i = V(\mathcal{A}_{i+1}^{(b)})$ and $B_i = \{u_0, \ldots, u_{b-1}\}$.

Observations

(i) $\mathcal{H}_i$ is i-uniform with $V(\mathcal{H}_i) = V(\mathcal{H}_{i+1})$.

(ii) $I \in \mathcal{I}(\mathcal{H}_{i+1}) \implies I \in \mathcal{I}(\mathcal{H}_i)$ (since no "forbidden" edge can be completely contained in I).

(iii) $B_i \subseteq I \subseteq A_i \cup B_i$.

(iv) The hypergraph $\mathcal{H}_i$ and the set A_i depend only on $\mathcal{H}_{i+1}$ and the set B_i (in particular, given B_i, we don't need to know the input I).

(v) Suppose that on inputs $(\mathcal{H}_{i+1}, I)$ and $(\mathcal{H}_{i+1}, I')$, the algorithm outputs $(A_i, B_i, \mathcal{H}_i)$ and $(A_i', B_i', \mathcal{H}_i')$ respectively. If $B_i \subseteq I'$ and $B_i' \subseteq I$, then $(A_i, B_i, \mathcal{H}_i) = (A_i', B_i', \mathcal{H}_i')$. This property will be used to prove "consistency" of fingerprints.

Algorithm for HCT II (Input: $\mathcal{H}$ satisfying assumptions of HCT II, $I \in \mathcal{I}(\mathcal{H})$). Let $\delta = (ck2^{k+1})^{-k}$. Set $i = k-1$, and repeat the following:

1. Apply the Scythe algorithm to $\mathcal{H}_{i+1}$ and I. Suppose it outputs $(A_i, B_i, \mathcal{H}_i)$.

2. If $|A_i| \leq (1-\delta)v(\mathcal{H})$, then set $q = i$, $r = i+1$ and STOP.

3. If $i > 1$, then set $i = i - 1$. Otherwise, set $q = r = 1$ and STOP.

Let's now define g_0 and f_0 as in the conclusion of HCT II: If $r > 1$, then the above algorithm must have stopped in Step 2 with some $|A_q| \leq (1-\delta)v(\mathcal{H})$. In this case, define $g_0(I) = B_{k-1} \cup \cdots \cup B_q$ and $f_0^*(I) = A_q$. If $r = 1$, then set $g_0(I) = B_{k-1} \cup \cdots \cup B_1$ and $f_0^*(I) = \{v \in V(\mathcal{H}_1) : \{v\} \notin \mathcal{H}_1\}$.

Observations

(i) $|g_0(I)| \leq (k-1)pv(\mathcal{H})$ for every $I \in \mathcal{I}(\mathcal{H})$

(ii) $g_0(I) \subseteq I \subseteq g_0(I) \cup f_0^*(I)$

(iii) Suppose for some $I, I' \in \mathcal{I}(\mathcal{H})$, $g_0(I) \subseteq I'$ and $g_0(I') \subseteq I$. Then, $g_0(I) = g_0(I')$ and $f_0^*(I) = f_0^*(I')$: Suppose we obtain the sequence $(B_{k-1}, \ldots, B_q)$ while running the algorithm on I, and the sequence $(B'_{k-1}, \ldots, B'_{q'})$ while running the algorithm on I'. Then, $B_i \subseteq g_0(I) \subseteq I'$ and $B'_i \subseteq g_0(I') \subseteq I$ for all i. Since $\mathcal{H}_k = \mathcal{H}'_k = \mathcal{H}$, it follows by repeatedly applying observation (v) from the Scythe algorithm that $B_i = B'_i$ for all i.

Next time, we will show that the following properties $(P3), (P4)$ are satisfied for all $\mathcal{H}_i$ during the execution of the above algorithm:

$(P3)$ $\Delta_\ell(\mathcal{H}_i) \leq \Delta_\ell^i$ for all $\ell \in [i]$, where $\Delta_\ell^i := \max\{2.\Delta_{\ell+1}^{i+1}, p.\Delta_\ell^{i+1}\}$ and $\Delta_\ell^k := \Delta_\ell(\mathcal{H})$. The removals we do in Step 4 of the scythe algorithm are to ensure that this holds. Also, by unwrapping the definition, one can show that $\Delta_1^i \leq c2^k p^{k-i} e(\mathcal{H})/v(\mathcal{H})$.

$(P4)$ $e(\mathcal{H}_i) \geq c_i p^{k-1} e(\mathcal{H})$ where $c_i = (ck2^{k+1})^{i-k}$. This will follow since we choose vertices in max-degree order, and hence add sufficiently many "forbidden" edges at each step.

Given these properties, let us show how to conclude:

(iv) $|f_0^*(I)| \leq (1-\delta)v(\mathcal{H})$ for all I: If $r > 1$, then this is true by construction. On the other hand, if $r = 1$, then since $\Delta_1(\mathcal{H}_1) \leq \Delta_1^1 \leq c2^k p^{k-1} e(\mathcal{H})/v(\mathcal{H})$ by $(P3)$, we have:

$$\begin{aligned}
|\{v \in V(\mathcal{H}_1) : \{v\} \in \mathcal{H}_1\}| &\geq \frac{e(\mathcal{H}_1)}{\Delta_1(\mathcal{H}_1)} \\
&\geq \frac{c_1 p^{k-1} e(\mathcal{H})}{\Delta_1(\mathcal{H}_1)} \\
&\geq \frac{c_1 p^{k-1} e(\mathcal{H})}{c 2^k p^{k-1} e(\mathcal{H})/v(\mathcal{H})} \\
&= \frac{(ck2^{k+1})^{1-k}}{c2^{k-1}} v(\mathcal{H}) \\
&\geq \delta v(\mathcal{H}),
\end{aligned}$$

where the second inequality uses $(P4)$, the third inequality uses the above bound on $\Delta(\mathcal{H}_1)$ coming from $(P3)$, and the last inequality uses $\delta = (ck2^{k+1})^{-k}$.

This finishes the proof modulo proving that $(P3)$ and $(P4)$ hold.

To finish for today, let's discuss the intuition between the assumption $\Delta_\ell(\mathcal{H}) \leq cp^{\ell-1} e(\mathcal{H})/v(\mathcal{H})$.

(i) If we have too many isolated vertices, then we also have a large independent set. Therefore, we want to ensure that at least $\approx \delta v(\mathcal{H})$ vertices are not isolated. This amounts to the condition

$$\frac{ke(\mathcal{H})}{\Delta_1(\mathcal{H})} \geq\approx \delta v(\mathcal{H}),$$

or equivalently, $\Delta_1(\mathcal{H}) \leq c\frac{e(\mathcal{H})}{v(\mathcal{H})}$ for some constant c.

(ii) To see the conditions taking p into account, consider selecting a random subset S of size $pv(\mathcal{H})$ from the independent set I to be the fingerprint. We hope that the number of edges of $\mathcal{H}$ that contain at least $k-1$ vertices of S is $\Omega(v(\mathcal{H}))$, since this is how the fingerprint ends up being useful. This translates to the condition

$$\frac{e(\mathcal{H})}{\Delta_k(\mathcal{H})} p^{k-1} = \Omega(v(\mathcal{H}))$$

i.e. $\Delta_k(\mathcal{H}) \leq cp^{k-1} e(\mathcal{H})/v(\mathcal{H})$.

(iii) Continuing with the same setup as before, imagine a situation where given a vertex v, there is a small collection of ℓ-sets ($2 \leq \ell \leq k-1$) such that every edge covering a vertex v contains one of these ℓ-sets. Note that the size of this collection is at least $\Omega(\deg_{\mathcal{H}}(v)/\Delta_\ell(\mathcal{H}))$, and by a small collection, we mean that it is actually $\Theta(\deg_{\mathcal{H}}(v)/\Delta_\ell(\mathcal{H}))$. In order to exclude v, S should contain at least $\ell-1$ elements in at least one of these ℓ-sets. Since the expected number of such ℓ-sets that have $\geq \ell - 1$ elements common with S is $\Theta(\deg_{\mathcal{H}}(v)p^{\ell-1}/\Delta_\ell(\mathcal{H}))$, it follows by summing over v that we should have

$$\frac{e(\mathcal{H})p^{\ell-1}}{\Delta_\ell(\mathcal{H})} = \Omega(v(\mathcal{H}))$$

i.e. $\Delta_\ell(\mathcal{H}) \leq cp^{\ell-1}e(\mathcal{H})/v(\mathcal{H})$

(iv) Note that while this heuristic gives good intuition for the parameters, (ii) and (iii) are not quite accurate, since there are other mechanisms through which a fingerprint may be useful. For instance, consider the disjoint union of the complete k-uniform hypergraph on n vertices and the empty k-uniform hypergraph on n vertices. Then, the fingerprint for an independent set I in the empty graph does not work by producing any "forbidden" edges. Rather, when we process vertices in maximum degree order, we skip all the vertices in the complete part before finding a vertex which is present in I.

April 27, 2018.

Last week, we sketched how to prove the following theorem:
Hypergraph containers theorem II: Let $k \in \mathbb{N}$ be fixed. For all $c > 0$, there exists $\delta > 0$ (we can take $\delta = (ck2^{k+1})^{-k}$) for which the following holds. Let $\mathcal{H}$ be a k-uniform hypergraph, and let $\mathcal{F}$ be the increasing family of sets of size at least $(1-\delta)v(\mathcal{H})$. Suppose $p \in (0,1)$ is such that for all $\ell \in [k]$,

$$\Delta_\ell(\mathcal{H}) \leq cp^{\ell-1}.\frac{e(\mathcal{H})}{v(\mathcal{H})}.$$

Then, there exists a family $\mathcal{S} \subseteq \binom{V(\mathcal{H})}{(k-1).pv(\mathcal{H})}$ and functions $f\colon \mathcal{S} \to \overline{\mathcal{F}}$ and $g\colon \mathcal{I}(\mathcal{H}) \to \mathcal{S}$ such that for all $I \in \mathcal{I}(\mathcal{H})$, $g(I) \subseteq I$ and $I \backslash g(I) \subseteq f(g(I))$. Moreover, if for some $I, I' \in \mathcal{I}(\mathcal{H})$, $g(I) \subseteq I'$ and $g(I') \subseteq I$, then $g(I) = g(I')$.

The proof is based on the so-called Scythe algorithm, and we completed the proof modulo the following two claims:
$(P3)$ $\Delta_\ell(\mathcal{H}_i) \leq \Delta_\ell^i$ for all $\ell \in [i]$, where $\Delta_\ell^i := \max\{2.\Delta_{\ell+1}^{i+1}, p.\Delta_\ell^{i+1}\}$ and $\Delta_\ell^k := \Delta_\ell(\mathcal{H})$.
Intuitively, this property holds because of the step in the Scythe algorithm where we remove all edges containing "dangerous" subsets.
$(P4)$ $e(\mathcal{H}_i) \geq c_i p^{k-i} e(\mathcal{H})$ where $c_i = (ck2^{k+1})^{i-k}$ i.e. $e(\mathcal{H}_i) \geq \left(\frac{p}{ck2^{k+1}}\right)^{k-i} e(\mathcal{H})$
Intuitively, this property holds because we choose vertices in the independent set which have high degree.

Remark: We also claimed that $\Delta_1^i \leq c2^k p^{k-i} e(\mathcal{H})/v(\mathcal{H})$, but this is easy to see from the definition above. Indeed,

$$\Delta_1^i \leq \max_{0 \leq d \leq k-i} \left\{2^d p^{k-i-d} \Delta_{d+1}(\mathcal{H})\right\} \leq \max_{0 \leq d \leq k-i} \left\{2^d p^{k-i-d} cp^d \frac{e(\mathcal{H})}{v(\mathcal{H})}\right\} \leq c.2^k p^{k-i} \frac{e(\mathcal{H})}{v(\mathcal{H})}.$$

In order to prove $(P3)$ and $(P4)$, we begin with some notation. For an i-uniform hypergraph $\mathcal{G}$ and $\ell \in [i]$, let

$$M_\ell^i(\mathcal{G}) := \left\{T \in \binom{V(\mathcal{G})}{\ell} : \deg_{\mathcal{G}}(T) \geq \frac{\Delta_\ell^i}{2}\right\}$$

denote the collection of "dangerous" ℓ-subsets. Now, we can complete our description of the Scythe algorithm from last time.

Scythe algorithm (Input: $\mathcal{H}_{i+1}$, $I \in \mathcal{I}(\mathcal{H}_{i+1})$). Let $\mathcal{A}_{i+1}^{(0)} = \mathcal{H}_{i+1}$ be the initial set of "available edges". Let $\mathcal{H}_i^{(0)}$ be the empty i-uniform hypergraph on the vertex set $V(\mathcal{H}_{i+1})$; this is the initial set of "forbidden edges". For $j = 0, 1, \ldots, b-1$, where $b = pv(\mathcal{H})$, do the following:

1. Let u_j be the first vertex of I in the max degree order on $V(\mathcal{A}_{i+1}^{(j)})$. If $I \cap V(\mathcal{A}_{i+1}^{(j)}) = \emptyset$ (think of this as "atypical" for the algorithm), then STOP with $\mathcal{H}_i = \mathcal{H}_i^{(0)}$, $A_i = \emptyset, B_i = \{u_0, \ldots, u_{j-1}\}$.

2. Add the edges incident to u_j to $\mathcal{H}_i^{(j)}$ in order to get $\mathcal{H}_i^{(j+1)}$.

3. Let $\mathcal{A}_{i+1}^{(j+1)}$ be the induced hypergraph on the vertex set $V(\mathcal{A}_{i+1}^{(j)}) \backslash W(u_j)$, where $W(u_j)$ is the set of all vertices which preceded u_j in the max degree order (so we know that these vertices are not in I) along with u_j.

4. **Remove from $\mathcal{A}_{i+1}^{(j+1)}$ all edges which contain any subset in** $\cup_{\ell=1}^{i} M_\ell^i(\mathcal{H}_i^{(j+1)})$.

If we have not already stopped i.e. if the above loop executes b times, then set $\mathcal{H}_i = \mathcal{H}_i^{(b)}$, $A_i = V(\mathcal{A}_{i+1}^{(b)})$ and $B_i = \{u_0, \ldots, u_{b-1}\}$.

It's relatively easy to see that $(P3)$ holds inductively i.e. if $\Delta_{\ell+1}(\mathcal{H}_{i+1}) \leq \Delta_{\ell+1}^{i+1}$ for some $\ell \in [i]$, then $\Delta_\ell(\mathcal{H}_i) \leq \Delta_\ell^i$.

(i) If $\deg_{\mathcal{H}_i^{(j)}}(T) \geq \frac{\Delta_\ell^i}{2}$ for some $T \in \binom{V(\mathcal{H})}{\ell}$ and $j \in [b]$, then all edges containing T are removed from $\mathcal{A}_{i+1}^{(j)}$, and hence no further such edges can be added to $\mathcal{H}_i$. Therefore, $\deg_{\mathcal{H}_i}(T) = \deg_{\mathcal{H}_i^{(j)}}(T)$.

(ii) So, the only danger is that when we went from $\mathcal{H}_i^{(j-1)}$ to $\mathcal{H}_i^{(j)}$, we already added too many edges containing T. But this cannot happen either: when going from $\mathcal{H}_i^{(j-1)}$ to $\mathcal{H}_i^{(j)}$, we add all i-subsets D which, together with $\{u_j\}$, form an $(i+1)$-edge in $\mathcal{A}_{i+1}^{(j-1)}$. Therefore, if T contains u_j, then no edges containing T are added from $\mathcal{H}_i^{(j-1)}$ to $\mathcal{H}_i^{(j)}$. On the other hand, if T does not contain u_j, then

$$\deg_{\mathcal{H}_i^{(j)}}(T) - \deg_{\mathcal{H}_i^{(j-1)}}(T) \leq \deg_{\mathcal{H}_{i+1}}(T \cup \{u_j\}) \leq \Delta_{|T|+1}(\mathcal{H}_{i+1}).$$

Putting everything together, we get

$$\Delta_\ell(\mathcal{H}_i) \leq \frac{\Delta_\ell^i}{2} + \Delta_{\ell+1}(\mathcal{H}_{i+1}) \leq \frac{\Delta_\ell^i}{2} + \Delta_{\ell+1}^{i+1} \leq \Delta_\ell^i.$$

Finally, we will show that assuming $(P3)$ and $(P4)$ hold for $\mathcal{H}_{i+1}$, then either $(P4)$ holds for $\mathcal{H}_i$, or $|A_i| \leq (1-c_i)v(\mathcal{H})$ (in which case, we just stop). Recall that $c_i := (ck2^{k+1})^{i-k}$. In fact, we will show the stronger statement that if $|A_i| > (1-c_i)v(\mathcal{H})$, then we must have

$$e(\mathcal{H}_i) \geq \frac{p}{c2^{k+1}k} e(\mathcal{H}_{i+1}).$$

To set this up, let's start with a preliminary computation. Let $\tilde{\mathcal{A}}_{i+1}^{(j)}$ denote the subhypergraph induced by $\mathcal{A}_{i+1}^{(j)}$ on the vertex set $\{u_j\} \cup V(\mathcal{A}_{i+1}^{(j)}) \backslash W(u_j)$. Then,

$$\begin{aligned} e(\mathcal{H}_i^{(j+1)}) - e(\mathcal{H}_i^{(j)}) &= \deg_{\mathcal{A}_{i+1}^{(j)}}(u_j) \\ &\geq \frac{(i+1)e(\tilde{\mathcal{A}}_{i+1}^{(j)})}{v(\tilde{\mathcal{A}}_{i+1}^{(j)})} \\ &\geq \frac{(i+1)e(\mathcal{A}_{i+1}^{(j+1)})}{v(\mathcal{H})} \end{aligned}$$

Case 0: If $(i+1)e(\mathcal{A}_{i+1}^{(j+1)}) \geq e(\mathcal{H}_{i+1})$ for every $j \in \{0, \ldots, b-1\}$, then

$$e(\mathcal{H}_i) \geq \sum_{j=0}^{b-1} \frac{(i+1)e(\mathcal{A}_{i+1}^{(j+1)})}{v(\mathcal{H})} \geq b.\frac{e(\mathcal{H}_{i+1})}{v(\mathcal{H})} = p.e(\mathcal{H}_{i+1}) \geq \frac{p}{2^{k+1}k} e(\mathcal{H}_{i+1})$$

and we are done.

So, we may assume that for some j,

$$e(\mathcal{A}_{i+1}^{(b)}) \leq e(\mathcal{A}_{i+1}^{(j+1)}) < \frac{e(\mathcal{H}_{i+1})}{i+1}.$$

In particular, we are assuming that during the execution of the Scythe algorithm, many edges are removed from $\mathcal{H}_{i+1}$. We will show that this happens only if either many of the $W(u_j)$'s are large in size, or some family $M_\ell^i(\mathcal{H}_i)$ is large. The next computation makes this formal.

First, note that

$$\begin{aligned} e(\mathcal{A}_{i+1}^{(j)}) - e(\mathcal{A}_{i+1}^{(j+1)}) &\leq |W(u_j)|.\Delta_1(\mathcal{H}_{i+1}) + \sum_{\ell=1}^{i} \left| M_\ell^i(\mathcal{H}_i^{(j+1)} \backslash M_\ell^i(\mathcal{H}_i^{(j)}) \right| .\Delta_\ell(\mathcal{H}_{i+1}) \\ &\leq |W(u_j)|.\Delta_1^{i+1} + \sum_{\ell=1}^{i} \left| M_\ell^i(\mathcal{H}_i^{(j+1)} \backslash M_\ell^i(\mathcal{H}_i^{(j)}) \right| .\Delta_\ell^{i+1}, \end{aligned}$$

where the second inequality holds by our inductive assumption that $(P3)$ holds for $\mathcal{H}_{i+1}$. Summing this over j, we get

$$\frac{i}{i+1}e(\mathcal{H}_{i+1}) \leq e(\mathcal{H}_{i+1})-e(\mathcal{A}_{i+1}^{(b)}) \leq \sum_{j=0}^{b-1}|W(u_j)|.\Delta_1^{i+1}+\sum_{\ell=1}^{i}|M_\ell^i(\mathcal{H}_i^{(b)})|.\Delta_\ell^{i+1}$$

Therefore, if $\sum_{j=0}^{b-1}|W(u_j)|.\Delta_1^{i+1} < \frac{e(\mathcal{H}_{i+1})}{4} \leq \frac{i}{2(i+1)}.e(\mathcal{H}_{i+1})$, then we must have $\sum_{\ell=1}^{i}|M_\ell^i(\mathcal{H}_i^{(b)})|.\Delta_\ell^{i+1} \geq \frac{1}{2}\frac{i}{i+1}e(\mathcal{H}_{i+1})$, and hence $|M_\ell^i(\mathcal{H}_i^{(b)})| \geq \frac{1}{2(i+1)}.e(\mathcal{H}_{i+1})$ for some $\ell \in [i]$.

Case 1: $|M_\ell^i(\mathcal{H}_i)| \geq \frac{1}{2(i+1)\Delta_\ell^i}.e(\mathcal{H}_{i+1})$ for some $\ell \in [i]$. In this case

$$\begin{aligned}
e(\mathcal{H}_i) &= \sum_{T\in\binom{V(\mathcal{H})}{\ell}} \deg_{\mathcal{H}_i}(T)/\binom{i}{\ell} \\
&\geq \sum_{T\in M_\ell^i(\mathcal{H}_i)} \deg_{\mathcal{H}_i}(T)/\binom{i}{\ell} \\
&\geq |M_\ell^i(\mathcal{H}_i)|.\Delta_\ell^i/2\binom{i}{\ell} \\
&\geq \frac{\Delta_\ell^i}{2^{i+2}(i+1)\Delta_\ell^{i+1}}e(\mathcal{H}_{i+1}) \\
&\geq \frac{p}{2^{k+1}k}e(\mathcal{H}_{i+1})
\end{aligned}$$

since $\Delta_\ell^i \geq p\Delta_\ell^{i+1}$.

Case 2: $\sum_{j=0}^{b-1}|W(u_j)| \geq \frac{1}{4\Delta_1^{i+1}}.e(\mathcal{H}_{i+1})$. In this case, we will show that $|A_i| \leq (1-c_i)v(\mathcal{H})$. Indeed,

$$\begin{aligned}
v(\mathcal{H}) - |A_i| &= v(\mathcal{A}_{i+1}^{(0)}) - v(\mathcal{A}_{i+1}^{(b)}) \\
&= \sum_{j=0}^{b-1}|W(u_j)| \\
&\geq \frac{e(\mathcal{H}_{i+1})}{4\Delta_1^{i+1}} \\
&\geq \frac{e(\mathcal{H}_{i+1})}{4}.\left(\frac{v(\mathcal{H})}{e(\mathcal{H})}.\frac{p^{i+1-k}}{c2^k}\right) \\
&\geq \frac{c_{i+1}p^{k-(i+1)}e(\mathcal{H})}{4}.\left(\frac{v(\mathcal{H})}{e(\mathcal{H})}.\frac{p^{(i+1)-k}}{c2^k}\right) \\
&\geq \frac{c_{i+1}}{c2^{k+2}}v(\mathcal{H}) \\
&\geq c_i v(\mathcal{H}).
\end{aligned}$$