HOMEWORK 1

DUE 01/23 AT 7:00AM PST

(1) Consider a symmetric simple random walk starting from $S_0 = 0$. For $A, B \geq 0$, let

$$\tau_{(A,-B)} = \inf\{n \geq 0 : S_n = A \text{ or } S_n = -B\}.$$

Show that for any $A, B > 0$,

$$\mathbb{P}[\tau_{(A,-B)} < \infty] = 1.$$

(2) Consider $n + 1$ points on a circle labelled (counterclockwise) as $0, 1, \ldots, n$. Consider the symmetric simple random walk on this circle with $n + 1$ points starting at 0.

(a) Show that with probability 1, the walk will eventually visit all $n + 1$ points.

(b) Show that for any $k \in \{1, \ldots, n\}$, the probability that k is the last point visited by the walk is $1/n$.

(c) Let T denote the first time when the random walk has visited all the points. Compute $\mathbb{E}[T]$.

Hint: Let τ_i denote the first time that the walk has visited i distinct points and let τ_{i+1} denote the first time that the walk has visited $i+1$ points. Argue that $\mathbb{E}[\tau_{i+1} - \tau_i] = i$.

(3) Consider a symmetric simple random walk in k dimensions starting from $(0, 0, \ldots, 0)$. This walk is described by the following rule: if the current state is $(x_1, \ldots, x_k) \in \mathbb{Z}^k$, then the next state is $(x_1 \pm 1, x_2 \pm 1, \ldots, x_k \pm 1)$ and each of these 2^k possibilities occurs with probability 2^{-k}.

(a) What is the probability that the walk is at $(0, 0, 0, 0)$ at time ℓ?

(b) For $k = 1, 2, 3$ estimate (you may use a computer) the expected number of times that the walk is at $(0, \ldots, 0)$.

(4) Let $(S_n)_{n\geq 0}$ and $(S'_n)_{n\geq 0}$ be two independent symmetric simple random walks starting from $S_0 = 0 = S'_0$. For $j \in \mathbb{Z}$, let T_j denote the number of times that the two walks meet at j i.e.

$$T_j = |\{n \geq 0 : S_n = j = S'_n\}|.$$

What is $\mathbb{E}[T_j]$?

(5) Suppose Alice and Bob are playing a game in which Alice wins each round with probability p and Bob wins each round with probability $q = 1-p$. The results of different rounds are independent. The winner of the game is the player who first wins $2n + 1$ rounds.

(a) What is the probability that Alice wins the game in r rounds?

(b) What is the probability that the game ends in r rounds?

(c) $(*)$ Suppose that $p = q = 1/2$. Find the expected length of the game and use Stirling's approximation to estimate your result.

Hint: Let p_r denote the probability that the game ends in $4n + 1 - r$ rounds. Show that

$$(2n - r)p_r = \frac{1}{2}(4n + 1)p_{r+1} - \frac{1}{2}(r + 1)p_{r+1},$$

and use that $\sum_r p_r = 1$.

(6) Let $(S_n)_{n\geq 0}$ be a symmetric simple random walk starting at $S_0 = 0$. Show that

(a) $(*)$ $\mathbb{P}[S_1 \neq 0, S_2 \neq 0, \ldots, S_{2k} \neq 0] = \mathbb{P}[S_{2k} = 0]$.

(b) $\mathbb{P}[S_1 > 0, S_2 > 0, \ldots, S_{2k} > 0] = \frac{1}{2}\mathbb{P}[S_{2k} = 0]$.

(c) $(*)$ $\mathbb{P}[S_1 \geq 0, S_2 \geq 0, \ldots, S_{2k} \geq 0] = \mathbb{P}[S_{2k} = 0]$.

(7) Let $(S_n)_{n\geq 0}$ be a simple random walk for which each step is independently $+1$ with probability p and -1 with probability $q = 1-p$. Suppose that $S_0 = 0$. Show that:

(a) For any $k > 0$,
$$\mathbb{P}[S_1 > 0, \ldots, S_{n-1} > 0, S_n = k] = \frac{k}{n}\mathbb{P}[S_n = k].$$

(b) For any $k \neq 0$,
$$\mathbb{P}[S_1 \neq 0, \ldots, S_{n-1} \neq 0, S_n = k] = \frac{|k|}{n}\mathbb{P}[S_n = k].$$

(c) $\mathbb{P}[S_1 \neq 0, \ldots, S_n \neq 0] = \frac{\mathbb{E}[|S_n|]}{n}$.

(8) Let $(S_n)_{n\geq 0}$ be a symmetric simple random walk starting at $S_0 = 0$. For any integer x, let
$$\tau_x = \min\{n \geq 0 : S_n = x\}$$
be the first time to visit x and for any $n = 0, 1, 2, \ldots$ let
$$M_n = \max\{S_0, S_1, \ldots, S_n\}$$
be the maximum value of the walk until time n. Show that:

(a) For $x \geq 0$,
$$\mathbb{P}[M_m \geq x] = \mathbb{P}[\tau_x \leq m].$$

(b) For any $y \geq 0$ and for any x,
$$\mathbb{P}[M_n \geq y, S_n = x] = \begin{cases} \mathbb{P}[S_n = x] & \text{if } x \geq y, \\ \mathbb{P}[S_n = 2y - x] & \text{if } x < y. \end{cases}$$

Hint: If $x < y$, then reflect the path after the first time it hits y.

(c) For any $y \geq 0$,
$$\mathbb{P}[M_n \geq y] = \mathbb{P}[S_n = y] + 2\mathbb{P}[S_n > y].$$

(d) For any $y \geq 0$,
$$\mathbb{P}[M_n = y] = \max\{\mathbb{P}[S_n = y], \mathbb{P}[S_n = y+1]\}.$$

(9) $(*)$ Let $(S_n)_{n\geq 0}$ be a symmetric simple random walk starting at $S_0 = 0$. For $n = 0, 1, \ldots,$ let
$$M_n = \max\{S_0, S_1, \ldots, S_n\}$$
be the maximum value of the walk until time n. For $n \geq 1$, let
$$\tau_{2n} = \min\{0 \leq i \leq 2n : S_i = M_{2n}\}.$$
In words, τ_{2n} is the first time that the walk attains its maximum value in the first $2n$ steps. Show that:

(a) $\mathbb{P}[\tau_{2n} = 0] = \mathbb{P}[S_{2n} = 0]$.

(b) $\mathbb{P}[\tau_{2n} = 2n] = \frac{1}{2}\mathbb{P}[S_{2n} = 0]$.

(c) For any $0 < k < 2n$, writing $k = 2m$ or $k = 2m+1$,
$$\mathbb{P}[\tau_{2n} = k] = \frac{1}{2}\mathbb{P}[S_{2m} = 0]\mathbb{P}[S_{2n-2m} = 0].$$

Hence, for $1 \leq m \leq n-1$,
$$\mathbb{P}[\tau_{2n} = 2m \text{ or } \tau_{2n} = 2m+1] = \mathbb{P}[S_{2m} = 0]\mathbb{P}[S_{2n-2m} = 0].$$

Hint: Use time reversal along with the results of Problem 6.