HOMEWORK 3

DUE 02/06 AT 7:00PM PST

(1) Let $X_1, \dots, X_n$ be independent random variables with $X_i \sim \text{Exp}(\lambda_i)$.
 (a) Show that

$$\max\{X_1, \dots, X_n\} = \sum_{k=1}^{n} (-1)^{k-1} \sum_{i_1, \dots, i_k \text{ distinct}} \min\{X_{i_1}, \dots, X_{i_k}\}.$$

 (This holds as an equality on the level of real numbers, but if you're not able to show that, it's fine to show that the random variables on either side of = have the same distribution).

 (b) Use this to find

$$\mathbb{E}[\max\{X_1, \dots, X_n\}].$$

(2) (due to Durrett) Messages arrive to be transmitted across the internet at arrival times of a Poisson process with rate λ. Let Y_i be the size of the ith message, and suppose that $Y_1, Y_2, \dots$ are i.i.d. with common generating function $g(z) = \mathbb{E}[z^{Y_i}]$. As usual, let $N(t)$ denote the number of arrivals by time t and let $S(t) = Y_1 + \cdots + Y_{N(t)}$ be the total size of the messages that have arrived up to time t.
 (a) Find the generating function $f_t(z) = \mathbb{E}[z^{S(t)}]$.
 (b) Find $\mathbb{E}[S(t)]$ and $\text{Var}(S(t))$.

(3) Consider red and blue points scattered on $[0, \infty)$ according to independent Poisson point processes with rates λ_R and λ_B respectively. Consider the red or blue points whose nearest neighbor to the right is blue. Show that these points are distributed according to a PPP on $[0, \infty)$ with rate λ and compute λ.

(4) (due to Durrett) Let $N_1(t)$ and $N_2(t)$ denote two independent Poisson processes with rates λ_1 and λ_2. Fix non-negative integers i and j. What is the probability that the two dimensional process $(N_1(t), N_2(t))$ will ever hit (i, j)?

(5) (due to Durrett) Two copy editors read a 300 page manuscript. The first found 100 typos, the second found 120 typos, and their lists contain 80 typos in common. Suppose that the author's typos follow a Poisson process with some unknown rate λ per page and that the two copy editors catch each error independently with unknown probabilities of success p_1 and p_2. Let X_0 denote the number of typos that neither found, X_1 denote the number of typos that only copy editor 1 found, X_2 denote the number of typos that only copy editor 2 found, and X_3 denote the number of typos found by both.
 (a) Find the joint distribution of (X_0, X_1, X_2, X_3).
 (b) Use this to find estimates of p_1 and p_2 and then, of the number of undiscovered typos.

(6) $(*)$ (due to Durrett) Consider a Poisson point process with rate λ and arrival times $T_1, T_2, \dots$. Let $c > 0$ and let

$$J = \min\{n \geq 1 : T_n - T_{n-1} > c\},$$

where $T_0 = 0$. Show that

$$\mathbb{E}[T_{J-1} + c] = \frac{e^{\lambda c} - 1}{\lambda}.$$

(7) (due to Pinsky and Karlin) Let $U_1, U_2, \ldots$ be independent random variables, each uniformly distributed over the interval $[0,1]$. These random variables represent successive bids on an asset that you are trying to sell. You must sell the asset by time 1, otherwise, it becomes worthless. The bids arrive according to a Poisson point process on $[0,\infty)$ with rate 1.

(a) Let $\theta \in (0,1)$. Suppose your strategy is to accept the first offer which is strictly greater than θ. What is the probability that you will sell the asset by time 1?

(b) What is the expected value of the bid that you accept? What is the value of θ that maximizes your expected return?

(c) Suppose that you instead follow the strategy of accepting the first offer above $\theta(t)$ where

$$\theta(t) = \frac{1-t}{3-t}.$$

What is the probability of selling the asset by time 1? What is your expected revenue?

(8) $(*)$ You are tasked with estimating the mean of an exponentially distributed random variable ξ. The data that is available to you is a Poisson point process $N(t)$ on $[0,t_*]$ whose interarrival times are i.i.d. copies of ξ. Here, $t_* > 0$ (which may be interpreted as the duration of your experiment) is fixed in advance. Suppose that the interarrival times of this PPP are $W_1, \ldots, W_{N(t_*)}$. Then, it is natural to try to estimate $\mu = \mathbb{E}[\xi]$ by

$$\hat{\mu} := \frac{W_1 + \cdots + W_{N(t_*)}}{N(t_*)}.$$

Show that

$$\frac{\mu - \mathbb{E}[\hat{\mu} \mid N(t_*) > 0]}{\mu} = \frac{\mathbb{E}[N(t_*)]}{e^{\mathbb{E}[N(t_*)]} - 1}.$$

In particular, on average, $\hat{\mu}$ underestimates μ.

(9) The number of incoming calls to a call center is distributed according to an inhomogeneous PPP with rate

$$\lambda(t) = 24\sin(t) + 24,$$

where the unit of t is $\frac{1}{2\pi}$ days. Thus, the first day last from $t = 0$ to $t = 2\pi$, the second day lasts from $t = 2\pi$ to $t = 4\pi$ and so on.

(a) How many calls, in expectation, does the call center receive during a given day?

(b) Suppose that each call takes exactly 30 minutes. Let T be the total time during a day that exactly 3 calls are being handled by the call center. What is $\mathbb{E}[T]$?

(10) $(*)$ Let $W_1, W_2, \ldots$ denote i.i.d. Exp(1) random variables. Let $t_* > 0$ be fixed and let k be such that

$$W_1 + \cdots + W_{k-1} < t_* \leq W_1 + \cdots + W_k.$$

Note that k is random. Let $f(x)$ denote the pdf of W_k. Show that

$$f(x) = \begin{cases} xe^{-x} & \text{for} \quad 0 < x \leq t_* \\ (1+t_*)e^{-x} & \text{for} \quad x > t_*. \end{cases}$$

In particular, show that

$$\lim_{t_* \to \infty} \mathbb{E}[W_k] = 2.$$

(What is going on here?)