HOMEWORK 4

DUE 02/13 AT 7:00PM PST

(1) Consider a DTMC on the state space $\{1, 2, \dots, k\}$ with transition matrix P satisfying

$$\sum_{i=1}^{k} P_{ij} = 1 \quad \forall j = 1, \dots, k.$$

Suppose that $X_0 \sim \mathrm{Unif}(\{1, \dots, k\})$. Show that, for all $n \geq 0$, $X_n \sim \mathrm{Unif}(\{1, \dots, k\})$.

(2) (due to Durrett) Suppose that the weather on a given day is either classified as "rainy" or "sunny". Suppose that the probability that it is rainy today is 0.3 if neither of the last two days was rainy, and 0.6 if at least one of the last two days was rainy. What is the probability that it will be rainy on Wednesday given that it was sunny on both Sunday and Monday?

(3) (a) Let $(X_n)_{n\geq 0}$ be a DTMC on S with transition matrix P and let $(Y_n)_{n\geq 0}$ be an independent DTMC on S' with transition matrix P'. Is $Z_n = (X_n, Y_n)$ always a DTMC? If yes, what is the transition matrix? If not, provide an explanation.

(b) Let $(X_n)_{n\geq 0}$ be a DTMC on S with transition matrix P and let $(Y_n)_{n\geq 0}$ be an independent DTMC on S with transition matrix P'. Is $W_n = X_n + Y_n$ always a DTMC? If yes, what is the transition matrix? If not, provide an explanation.

(4) (a) Let $(X_n)_{n\geq 0}$ be a DTMC on S with initial distribution μ_0 and transition matrix P. Fix $k \in \mathbb{N}$. Show that $(X_{kn})_{n\geq 0}$ is a Markov chain. What are its initial distribution and transition matrix?

(b) (due to Grimmett and Stirzaker) Let $(Z_n)_{n\geq 1}$ be a sequence of i.i.d. throws of a fair six-sided die. For $n \geq 0$, let X_n denote the number of rolls (counted from Z_{n+1} onwards) until the next 6. Is $(X_n)_{n\geq 0}$ a DTMC? If yes, what is the transition matrix? If not, provide an explanation.

(5) ($*$) (due to Feller) Let $(Z_n)_{n\geq 0}$ be a sequence of i.i.d. throws of a fair six-sided die. Define $(X_n)_{n\geq 0}$ by

$$X_n = \max\{Z_0, \dots, Z_n\}.$$

Show that $(X_n)_{n\geq 0}$ is a DTMC. Calculate its transition matrix P and also calculate P^k for arbitrary $k = 1, 2, 3, \dots$.

(6) (due to Ross) Let $(X_n)_{n\geq 0}$ be a DTMC on S. Show that for all $N \in \mathbb{N}$ and for all $0 < k < N$,

$$\mathbb{P}[X_k = i_k \mid X_j = i_j \quad \forall \quad 0 \leq j \leq N, j \neq k] = \mathbb{P}[X_k = i_k \mid X_{k-1} = i_{k-1}, X_{k+1} = i_{k+1}].$$

(7) ($*$) (due to Levin and Peres) Consider the d-color Polya urn: initially, the urn contains one ball of each of d distinct colors. At each unit of time, a ball is selected uniformly at random from the urn and replaced along with an additional ball of the same color. Let N_t^i denote the number of balls in the urn of color i after t steps. Show that if $\boldsymbol{N}_t := (N_t^1, \dots, N_t^d)$, then $\boldsymbol{N}_t$ is uniformly distributed over the set

$$V_t = \left\{ (x_1, \dots, x_d) \in \mathbb{Z}^d, \quad x_i \geq 1 \quad \forall i = 1, \dots, d, \quad \sum_{i=1}^{d} x_i = t + d \right\}.$$

In particular, for $d = 2$, N_t^1 is uniformly distributed on $\{1, \dots, t+1\}$.

(8) (due to Durrett) For each of the following transition matrices, identify the communicating classes and determine which of them are transient and which of them are recurrent.

(a) $\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0.2 & 0 & 0.8 & 0 \\ 0.1 & 0.2 & 0.3 & 0.4 & 0 \\ 0 & 0.6 & 0 & 0.4 & 0 \\ 0.3 & 0 & 0 & 0 & 0.7 \end{bmatrix} \end{array}$

(b) $\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{bmatrix} 2/3 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 & 0 \end{bmatrix} \end{array}$

(9) ($*$) (A Hidden Markov Model) Suppose that the weather on a given day is either classified as "rainy" (R) or "sunny" (S). Suppose that the atmospheric pressure on a given day is either classified as "low" (L) or "high" (H). The probability that the atmospheric pressure changes its classification from one day to the next is 0.2. On a high pressure day, the probability that the day will be rainy is 0.25 whereas on a low pressure day, the probability that the day will be rainy is 0.75. Assume that the probability that today is a high pressure day is 0.5.

(a) Show that the classification of the atmospheric pressure is a DTMC. Let P be its transition matrix. Find P^n for all $n \geq 1$.

(b) Use this to find the covariance between the event that today is rainy and the event that it will be rainy n days from today.