HOMEWORK 6

DUE 02/27 AT 7:00PM PST

(1) Consider the two-state Markov chain with transition matrix

$$P = \begin{bmatrix} & A & B \\ A & 1-p & p \\ B & q & 1-q \end{bmatrix}$$

where $p, q \in (0,1)$. Show that

$$\mathbb{P}[X_n = A] = \frac{q}{p+q} + (1-p-q)^n \left(\mathbb{P}[X_0 = A] - \frac{q}{p+q} \right).$$

Thus, since $|1-p-q| < 1$, $\mathbb{P}[X_n = A]$ converges exponentially fast to the limiting value $\pi(A) = q/(p+q)$, where π is the unique stationary distribution.

(2) Consider the Gambler's ruin problem in which a gambler, who starts with K dollars, makes a sequence of independent bets in which she wins 1 dollar with probability p and loses 1 dollar with probability $1-p$. The gambler leaves the casino whenever her wealth hits a target N $(N > K)$ or she goes broke. Assume that $K > 1$. Find the probability of the following event: the gambler leaves the casino with N dollars and at some time before she leaves, her total wealth is 1 dollar.

(3) Consider a standard 8×8 chessboard, and recall that the king can move to any of the adjacent squares (if the king is in the middle of the board, there are 8 options; if the king is in a corner, there are 3 options, and so on). Consider the random walk on the chessboard in which, at every time step, one of the available legal moves for the king is chosen uniformly at random.

(a) Find the stationary distribution of this random walk.

(b) If the king starts in the bottom left corner, what is the expected number of moves for the king to first return to the bottom left corner?

(4) Consider a Markov chain on a finite state space $\{1, \ldots, N\}$ where $\{1, \ldots, r\}$ are transient states and $\{r, \ldots, N\}$ are absorbing states. Recall that the transition matrix for such a chain is of the form

$$P = \begin{bmatrix} Q & R \\ 0 & I \end{bmatrix}$$

(a) Show that, given $v_{r+1}, \ldots, v_N \geq 0$ such that $v_{r+1} + \cdots + v_N = 1$, the vector

$$\pi = (0, 0, \ldots, 0, v_{r+1}, \ldots, v_N) \in \mathbb{R}^N$$

is a stationary distribution of P.

(b) Show that these are all the stationary distributions of P.

(5) (Time reversals) Let $(X_n)_{0 \leq n \leq N}$ be an irreducible Markov chain with finite state space S and transition matrix P. Define the time-reversed chain $(Y_n)_{0 \leq n \leq N}$ by $Y_n = X_{N-n}$ for all $0 \leq n \leq N$.

(a) Show that $(Y_n)_{0 \leq n \leq N}$ satisfies the Markov property i.e. for all $1 \leq k \leq N$ and for all $i_0, i_1, \ldots, i_k \in S$,

$$\mathbb{P}[Y_k = i_k \mid Y_{k-1} = i_{k-1}, \ldots, Y_0 = i_0] = \mathbb{P}[Y_k = i_k \mid Y_{k-1} = i_{k-1}].$$

(b) Suppose the unique stationary distribution of P is π and $X_0 \sim \pi$. Show that for any $a, b \in S$,

$$\mathbb{P}[Y_{n+1} = b \mid Y_n = a] = \frac{\pi(b)}{\pi(a)} P_{b,a}.$$

In particular, show that if P satisfies the detailed balance condition with respect to π, then

$$\mathbb{P}[Y_{n+1} = b \mid Y_n = a] = P_{a,b}.$$

For this reason, chains satisfying the detailed balance condition are called *(time) reversible.*

(6) (Metropolis chain with a general base) Let π be a probability distribution on a finite set S and let Ψ be an irreducible transition matrix on S. Consider the Markov chain with the following transition rule: when at state $x \in S$, generate a state y from $\Psi(x, \cdot)$. Then, move to y with probability

$$\min\left(1, \frac{\pi(y)\Psi_{y,x}}{\pi(x)\Psi_{x,y}}\right)$$

and remain at x with the complementary matrix. Denote the transition matrix of this chain by P. Show that P is reversible with respect to π.

(7) ($*$) Consider an undirected graph $G = (V, E)$ with vertex set V and edge set E. Let $w \colon E \to \mathbb{R}^{>0}$ be an assignment of positive weights to edges. The random walk on the weighted undirected graph is defined as follows: let the current state be x and let $N_G(x)$ denote the neighbors of x i.e. the set of vertices that share an edge with x. Then, the probability that the next state is y is 0 if $y \notin N_G(x)$ and $w(\{x.y\})/\sum_{z \in N_G(x)} w(\{x, z\})$ if $y \in N_G(x)$.

(a) Suppose that the graph $G = (V, E)$ is connected. Show that this chain is irreducible and find its stationary distribution.

(b) Consider an irreducible Markov chain on S with transition matrix P which is reversible with respect to its stationary distribution π. Argue that this can be viewed as a random walk on a weighted undirected graph with vertex set $|S|$.
Hint: Look at $w(\{x, y\}) = \pi_x P_{x,y}$. Where do you use reversibility?

(8) ($*$) In card shuffling, we are given a deck of n cards and want to sample from the uniform distribution π on the set S of all permutations of $\{1, \ldots, n\}$. Show that the following shuffling procedures correspond to irreducible, aperiodic Markov chains on S with the unique stationary distribution π.

(a) Random transposition shuffle: Pick two cards i and j uniformly at random with replacement, and switch cards i and j.

(b) Top-to-random shuffle: Take the top card and place it at one of the n positions in the deck chosen uniformly at random.
Hint: Show that the transition matrix is doubly-stochastic.

(c) Riffle shuffle: (i) Split the deck into two parts according to Binomial$(n, 1/2)$, (ii) hold one part in your left hand and the other part in your right hand with the bottom of each deck facing the table, (iii) merge the two parts by dropping cards in sequence, where if you have L cards in your left hand and R cards in your right hand at some point, then the probability that the next card comes from your left hand is $L/(L + R)$.

(9) ($*$) Let P be an irreducible, aperiodic transition matrix on S with $|S| = n$. Since P is irreducible and aperiodic, the Perron-Frobenius theorem asserts that we can index the eigenvalues of A as $\lambda_1, \ldots, \lambda_n$, where $\lambda_1 = 1$ and $|\lambda_i| < 1$ for all $i > 2$. Let π be the unique stationary distribution of P and suppose further that P is reversible with respect to π.

(a) Let D be the $|S| \times |S|$ diagonal matrix with $D(x, x) = \sqrt{\pi(x)}$. Show that DPD^{-1} is a symmetric matrix. Then, use the spectral theorem from linear algebra to argue that all the eigenvalues of P are real and that there is a basis of left-eigenvectors $v_1, \ldots, v_n$ of P such that v_i is a left-eigenvector for λ_i.

(b) Use this to provide a proof of the fundamental theorem for Markov chains in the reversible case i.e. to show that for any $x, y \in S$, $P^t_{x,y} \to \pi(y)$ as $t \to \infty$.