HOMEWORK 7

DUE 03/06 AT 7:00PM PST

(1) Let μ and ν be probability measures on a finite set Ω. Show that

$$\max_{A\subseteq\Omega} |\mu(A) - \nu(A)| = \frac{1}{2}\sum_{x\in\Omega} |\mu(x) - \nu(x)|.$$

(2) Let μ and ν be probability measures on a finite set Ω. Let $\mathcal{F} = \{f : \Omega \to \mathbb{R}$ such that $|f(x)| \leq 1\ \forall x \in \Omega\}$. Show that there exists some $f \in \mathcal{F}$ such that

$$\mathrm{TV}(\mu, \nu) = \frac{1}{2}\left(\sum_{x\in\Omega} f(x)\mu(x) - \sum_{x\in\Omega} f(x)\nu(x)\right).$$

(3) Let $\mu \geq \nu > 0$. Show that

$$\mathrm{TV}(\mathrm{Pois}(\mu), \mathrm{Pois}(\nu)) \leq \mu - \nu.$$

Hint: Recall that the sum of independent Poisson random variables is also Poisson.

(4) For $i = 1, \ldots, n$, let μ_i and ν_i be probability measures on the finite set Ω_i. Consider the measures μ and ν on $\Omega = \prod_{i=1}^n \Omega_i$ defined by $\mu := \mu_1 \times \cdots \times \mu_n$ and $\nu := \nu_1 \times \cdots \times \nu_n$. Recall this means that for any $x = (x_1, \ldots, x_n) \in \Omega$,

$$\mu(x) = \mu_1(x_1) \times \cdots \times \mu_n(x_n)$$

and similarly for ν. Show that

$$\mathrm{TV}(\mu, \nu) \leq \sum_{i=1}^n \mathrm{TV}(\mu_i, \nu_i).$$

(5) (a) Let $(X_t)_{t\geq 0}$ be a DTMC with an irreducible transition matrix P on S with stationary distribution π. Let

$$\Delta_x(n) = \mathrm{TV}(X_n \mid X_0 = x, \pi).$$

Show that $\Delta(n+1) \leq \Delta(n)$ for all integers $n \geq 0$.

(b) Let X and Y be random variables taking values in some finite set Ω. Let $h \colon \Omega \to \Omega'$ be a function. Consider the random variables $X' = h(X)$ and $Y' = h(Y)$. Show the data-processing inequality for total variation distance:

$$\mathrm{TV}(X', Y') \leq \mathrm{TV}(X, Y).$$

(6) Let P be an irreducible and aperiodic transition matrix on a finite state space S. Show that there exists a positive integer n_0 such that

$$\min_{x\in S, y\in S} P^n(x, y) > 0 \quad \forall n \geq n_0.$$

You may use the following fact: let A be any set of non-negative integers such that $\gcd(A) = 1$ and such that for all $a, b \in A$, $a + b \in A$. Then, A contains all but finitely many of the non-negative integers.

(7) $(*)$ Let μ and ν be probability distributions on a finite set Ω. Show that there exists a coupling (X, Y) of μ and ν such that

$$\mathbb{P}[X \neq Y] = \mathrm{TV}(\mu, \nu).$$

Hint: It might help to first show that $\sum_{x\in\Omega} \min\{\mu(x), \nu(x)\} = 1 - \mathrm{TV}(\mu, \nu)$.

(8) ($*$) (due to Durrett) On each request, the i^{th} of n possible books is the one chosen with probability p_i. To make it quicker to find the book the next time, the librarian moves the book to the left end of the shelf. Define the state at any time to be the sequence of books we see as we examine the shelf from left to right. Since all the books are distinct, this list is a permutation of the set $\{1, 2, \ldots, n\}$, i.e., each number if listed exactly once. Show that the unique stationary distribution is given by

$$\pi(i_1, \ldots, i_n) = p_{i_1} \cdot \frac{p_{i_2}}{1 - p_{i_1}} \cdot \frac{p_{i_3}}{1 - p_{i_1} - p_{i_2}} \cdots \frac{p_{i_n}}{1 - p_{i_1} - \cdots - p_{i_{n-1}}}.$$

(9) ($*$) Let P be an irreducible Markov chain on a finite state space S with unique stationary distribution π. Suppose that P has period $k \geq 1$. Show that there exists a decomposition of S into k disjoint subsets, denoted by $S_0, \ldots, S_{k-1}$, such that for each $i = 0, \ldots, k-1$, the restriction of P^k to the states in S_i gives an irreducible aperiodic transition matrix on S_i. Also, for each $i = 0, \ldots, k-1$, find the unique stationary distribution of P^k restricted to S_i in terms of π.