# STATS 217: Introduction to Stochastic Processes I

Lecture 1

# Course information

- **Instructor**: Vishesh Jain
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- **Final grade based entirely on 9 problem sets**. See "Grading" section of course website for policies and further details.
- **Course website**: `jainvishesh.github.io/STATS217_Winter2021.html`.
- There are also associated **Canvas** and **Gradescope** sites that you should be enrolled in.

# Symmetric simple random walk

- $X_1, X_2, \dots$ is a sequence of independent and identically distributed (i.i.d.) **Rademacher random variables** i.e.,

$$\mathbb{P}[X_i = 1] = \mathbb{P}[X_i = -1] = 1/2 \quad \forall i.$$

- **Interpretation:** a gambler places bets on the outcome of fair coin tosses. If the outcome is heads, she wins $1 and if the outcome is tails, she loses $1. $X_i$ records the payout to the gambler in the $i^{th}$ round.
- Denote the initial wealth of the gambler by $S_0$.
- So, after $n$ rounds of betting, the wealth of the gambler is

$$S_n := S_0 + X_1 + \cdots + X_n.$$

# Symmetric simple random walk

- **Question 1:** What is the probability that the gambler is up by $100 before being down by $100?
- **Question 2:** What is the probability that the gambler is up by $200 before being down by $100?
- **Question 3:** Suppose that the gambler stops playing once she is either up by $200 or down by $100. What is the expected number of rounds she plays?
- **Question 4:** Suppose that the gambler stops playing once she is down by $100. What is the probability that she stops? What is the expected number of rounds she plays?
- **Question 5:** How do these answers change if $\mathbb{P}[X_i = 1] = 0.49$?

# Hitting time

- Given integers $A > 0, B > 0$, let

$$\tau = \tau_{(A,-B)} := \min\{n \geq 0 : S_n = A \text{ or } S_n = -B\}.$$

- On the homework, you will show that for any $A > 0, B > 0$,

$$\mathbb{P}[\tau < \infty] = 1.$$

- For $-B \leq k \leq A$, define

$$f(k) := \mathbb{P}[S_\tau = A \mid S_0 = k].$$

- **Question 1:** $A = 100, B = 100$, find $f(0)$.
- **Question 2:** $A = 200, B = 100$, find $f(0)$.
- **Question 3:** $A = 200, B = 100$, find $\mathbb{E}[\tau \mid S_0 = 0]$.
- **Question 4:** "$A = \infty$", $B = 100$, find (i) $\mathbb{P}[\tau < \infty \mid S_0 = 0]$ and (ii) $\mathbb{E}[\tau \mid S_0 = 0]$.

# First step analysis

Recall

$$f(k) := \mathbb{P}[S_\tau = A \mid S_0 = k] \quad \forall -B \leq k \leq A.$$

- Clearly $f(A) = 1, f(-B) = 0$.
- For every $-B < k < A$,

$$\begin{aligned} f(k) &= \frac{1}{2} \cdot \mathbb{P}[S_\tau = A \mid S_0 = k, X_1 = 1] + \frac{1}{2} \cdot \mathbb{P}[S_\tau = A \mid S_0 = k, X_1 = -1] \\ &= \frac{1}{2} \cdot f(k+1) + \frac{1}{2} \cdot f(k-1). \end{aligned}$$

- Let $f(-B+1) = x$. Then, the above relation gives $f(-B+2) = 2x$. Similarly,

$$f(-B+\ell) = \ell x \quad \forall 0 \leq \ell \leq A+B.$$

- Since $f(A) = 1$, we must have

$$x = \frac{1}{A+B}.$$

# First step analysis

We have proved that

$$f(k) = \mathbb{P}[S_\tau = A \mid S_0 = k] = \frac{k+B}{A+B} \quad \forall -B \leq k \leq A.$$

- **Answer 1:** $A = 100, B = 100,\ f(0) = 1/2.$
- **Answer 2:** $A = 200, B = 100,\ f(0) = 1/3.$
- Another interpretation of this scenario is the following: suppose Alice and Bob bet on the outcomes of fair coin tosses. If the outcome is heads, then Bob pays \$1 to Alice, otherwise Alice pays \$1 to Bob. If Alice starts with \$$A$ and Bob starts with \$$B$ then the probability that Alice wins everything ('Alice ruins Bob') is

$$\frac{A}{A+B}.$$

# Application: symmetric simple random walk on the circle

Consider the symmetric simple random walk on the circle with $n+1$ points, starting from the point marked 0.

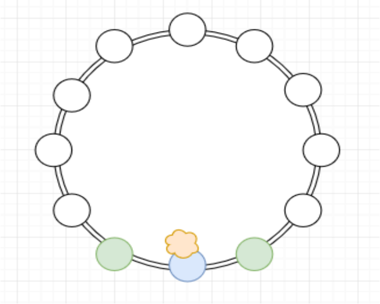

Image courtesy of user 'mark' on math.stackexchange.com

# Application: symmetric simple random walk on the circle

- Similar to the homework exercise, it follows that with probability 1, the random walk visits all points.
- Therefore, some point other than 0 is the last point visited.
- What is the probability that 1 is the last point visited?

$$\begin{aligned}\mathbb{P}[1 \text{ is the last point visited}] &= \mathbb{P}[2 \text{ is visited before } 1] \\ &= \mathbb{P}[S_{\tau_{(n-1,-1)}} = n-1 \mid S_0 = 0] \\ &= \frac{1}{n}.\end{aligned}$$

- On the homework, you will show that for all $1 \leq k \leq n$,

$$\mathbb{P}[k \text{ is the last point visited}] = \frac{1}{n}.$$

# First step analysis

- Given integers $A > 0, B > 0$, let

$$\tau := \min\{n \geq 0 : S_n = A \text{ or } S_n = -B\}.$$

- For $-B \leq k \leq A$, define

$$g(k) := \mathbb{E}[\tau \mid S_0 = k].$$

- Clearly, $g(-B) = 0, g(A) = 0$.
- For $-B < k < A$, we have

$$\begin{aligned} g(k) &= \frac{1}{2}\mathbb{E}[\tau \mid S_0 = k, X_1 = 1] + \frac{1}{2}\mathbb{E}[\tau \mid S_0 = k, X_1 = -1] \\ &= \frac{1}{2}\left(g(k+1) + 1\right) + \frac{1}{2}\left(g(k-1) + 1\right) \\ &= \frac{1}{2}g(k+1) + \frac{1}{2}g(k-1) + 1. \end{aligned}$$

# First step analysis

- Let $(\Delta h)(k) := h(k+1) - h(k)$.
- Then, for all $-B < k < A$

$$\begin{aligned}(\Delta(\Delta g))(k-1) &= (\Delta g)(k) - (\Delta g)(k-1) \\ &= g(k+1) - g(k) - g(k) + g(k-1) \\ &= g(k+1) - (g(k+1) + g(k-1) + 2) + g(k-1) \\ &= -2.\end{aligned}$$

- "Second derivative of $g$ is -2" so $g(k) = -k^2 + Dk + C$.
- Using boundary conditions,

$$g(k) = -(k-A)(k+B).$$

# First step analysis

We have proved that

$$g(k) = \mathbb{E}[\tau \mid S_0 = k] = -(k + A)(k - B).$$

- **Answer 3:** $A = 200, B = 100$, $g(0) = 2 \times 10^4$.
- **Answer 4 (ii):** "$A = \infty$", $B = 100$, $g(0) = \infty$.
- Formally, let

$$\tau_1 = \min\{n \geq 0 : S_n = -100\},$$
$$\tau_2(\ell) = \min\{n \geq 0 : S_n = -100 \text{ or } S_n = \ell\} \quad \forall \ell \geq 1.$$

- Then, for all $\ell \geq 1$, $\tau_2(\ell) \leq \tau_1$ so that

$$100\ell = \mathbb{E}[\tau_2(\ell) \mid S_0 = 0] \leq \mathbb{E}[\tau_1 \mid S_0 = 0],$$

and now take $\ell \to \infty$.

# First step analysis

- In words, for a symmetric simple random walk starting at 0, the expected time to hit $-100$ is infinite! Of course, there is nothing special about $-100$ here.
- On the other hand, **Answer 4(i):**

$$\begin{aligned}\mathbb{P}[S_n \text{ visits } -100] &\geq \mathbb{P}[S_{\tau_2(\ell)} = -100] \\ &= \frac{\ell}{100+\ell} \\ &\to 1 \text{ as } \ell \to \infty.\end{aligned}$$

- So, a symmetric simple random walk starting at 0 visits $-100$ with probability 1. Again, there is nothing special about $-100$ here.