STATS 217: Introduction to Stochastic Processes I

Lecture 10

Recurrence and transience

$f_s := \mathbb{P}(T_s < \infty)$

Let $(X_n)_{n\geq 0}$ be a DTMC on S.

$T_s := \tau_{\{s\},s}$

- $s \in S$ is a **recurrent state** if $f_s = 1$.
- $s \in S$ is a **transient state** if $f_s < 1$.

Recurrence and transience

Let $(X_n)_{n\geq 0}$ be a DTMC on S.

- $s \in S$ is a **recurrent state** if $f_s = 1$.
- $s \in S$ is a **transient state** if $f_s < 1$.

- By the formula

$$\mathbb{E}[N(s) \mid X_0 = s] = \frac{f_s}{1 - f_s},$$

  we see that

  - f_s is recurrent $\iff \mathbb{E}[N(s) \mid X_0 = s] = \infty$.
  - f_s if transient $\iff \mathbb{E}[N(s) \mid X_0 = s] < \infty$.

Accessibility

$f_s = f_{s \to s}$

Recall that for any $A \subset S, a \in S$

$$\tau_{A,a} = \min\{n \geq 1 : X_n \in A \mid X_0 = a\}.$$

For $a, b \in S$, we let

$$f_{a \to b} = \mathbb{P}[\tau_{\{b\},a} < \infty].$$

Accessibility

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$$f_{a \to b} = \mathbb{P}[\tau_{\{b\},a} < \infty].$$

- For $a, b \in S$, we say that b is **accessible** from a, denoted by $a \to b$, if at least one of the following holds: (i) $a = b$, (ii) $f_{a \to b} > 0$.

Accessibility

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$$f_{a \to b} = \mathbb{P}[\tau_{\{b\},a} < \infty].$$

- For $a, b \in S$, we say that b is **accessible** from a, denoted by $a \to b$, if at least one of the following holds: (i) $a = b$, (ii) $f_{a \to b} > 0$.
- Note that

$$a \to b \iff p^n_{a,b} > 0 \text{ for some } n \geq 0.$$

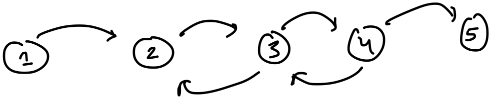
convention: $p^0_{a,b} = \mathbb{1}[a=b]$

$p^n_{ij} = \mathbb{P}[X_n = j \mid X_0 = i]$

say RHS w/ $n = 100$

Communication

Let $a, b \in S$, we say that a and b **communicate**, denoted by $a \leftrightarrow b$, if $a \to b$ and $b \to a$.



denote $p_{ij} > 0$

$2 \leftrightarrow 3$, $3 \leftrightarrow 4$ $\Rightarrow$ $2 \leftrightarrow 4$

$1 \leftrightarrow 1$

$5 \leftrightarrow 1$

step 1: only 1 comm. with 1

$S_1 = \{1\}$, $S_1' = \{2, 3, 4, 5\}$.

step 2: $\{2, 3, 4\}$ comm. with 2.

$\{1\} \longrightarrow \{2, 3, 4\} \longrightarrow \{5\}$

Communication

Let $a, b \in S$, we say that a and b **communicate**, denoted by $a \leftrightarrow b$, if $a \to b$ and $b \to a$.

Observe that communication is an equivalence relation i.e.,

this is a relation on $S \times S$ satisfying three props.

- Reflexive: $a \leftrightarrow a$ for all $a \in S$.
- Symmetric: $a \leftrightarrow b \implies b \leftrightarrow a$ for all $a, b \in S$.
- Transitive: $a \leftrightarrow b$ and $b \leftrightarrow c \implies a \leftrightarrow c$ for all $a, b, c \in S$.

$a \to b \quad b \to c \qquad a \overset{?}{\to} c$

$b \to a \quad c \to b \qquad c \overset{?}{\to} a$

$c \to b \to a$

Communication

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Therefore, we can partition S into **(maximal) communicating classes** i.e.

$$S = S_1 \cup \cdots \cup S_k, \quad \text{such that}$$

algorithm:
→ start w/ state 1
→ find everything that comm. w 1
→ call this S_1
→ $S' = S \setminus S_1$
repeat.

- $S_1, \ldots, S_k$ are disjoint.
- $a \leftrightarrow b$ for all $a, b \in S_i$, for all $i = 1, \ldots, k$.
- $a \not\leftrightarrow b$ for all $a \in S_i$, $b \in S_j$, $i \neq j$.

Class properties

Let $S = S_1 \cup \cdots \cup S_k$ denote the decomposition of S into (maximal) communicating classes.

motivation for name: equiv/comm. classes

- We say that a property is a **class property** if for every $i = 1, \ldots, k$, either all $s \in S_i$ have the property or no $s \in S_i$ have the property.

$\{1\}$ $\{2, 3, 4\}$ $\{5\}$

P is a class prop.

then either all of 2, 3, 4 have it

OR none of them have it.

Class properties

Let $S = S_1 \cup \cdots \cup S_k$ denote the decomposition of S into (maximal) communicating classes.

- We say that a property is a **class property** if for every $i = 1, \ldots, k$, either all $s \in S_i$ have the property or no $s \in S_i$ have the property.
- We will now show that recurrence is a class property.
- Note that this just means that if $\underbrace{a \text{ is recurrent}}_{f_{a\to a} = 1}$ and $a \leftrightarrow b$, then $\underbrace{b \text{ is recurrent}}_{f_{b\to b} = 1}$.

Class properties

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- We say that a property is a **class property** if for every $i = 1, \ldots, k$, either all $s \in S_i$ have the property or no $s \in S_i$ have the property.
- We will now show that recurrence is a class property.
- Note that this just means that if a is recurrent and $a \leftrightarrow b$, then b is recurrent.
- In fact, we will show something more, namely that

$$a \text{ is recurrent, and } a \to b \implies b \text{ is recurrent, and } b \to a.$$

$a \longrightarrow b$

if I tell you that a is recurrent, then you can add

We want to show that

$$a \text{ is recurrent and } a \to b \implies \overset{(2)}{b \text{ is recurrent}} \text{ and } \overset{(1)}{b \to a}.$$

Why is this true?

Accessibility and recurrence

We want to show that

$$a \text{ is recurrent and } a \to b \implies b \text{ is recurrent and } b \to a.$$

Why is this true? Intuitively,

- a being recurrent means that a returns to itself with probability 1. $a \to b$ means that there is a positive probability of going from a to b. If it were the case that $b \not\to a$, then once we get to b, we have no way of getting back to a, contradicting recurrence.

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- Now, to see the recurrence of b, since we visit a infinitely many times in expectation and since there is a positive probability of going from a to b and of going from b to a, we also visit b infinitely many times in expectation.

Accessibility and recurrence

wts: a is Recurrent
$a \longrightarrow b$ $\Longrightarrow$ $b \longrightarrow a$.

Formally, we have the inclusion of events

$$\{N(a) < \infty \mid X_0 = a\} \supseteq \{X_n \text{ visits } b \text{ and never returns to } a \mid X_0 = a\}.$$

take prob. on both sides

$\mathbb{P}$ [left event] = 0

$0 = \mathbb{P}$ [Right] $\geq \underbrace{f_{a\to b}}_{\substack{>0 \\ \text{(by assumption)}}} (\overline{1 - f_{b\to a}})$

Accessibility and recurrence

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Taking probabilities, we have

$$\begin{aligned} 0 &= \mathbb{P}[N(a) < \infty \mid X_0 = a] \\ &\geq f_{a\to b} \cdot (1 - f_{b\to a}). \end{aligned}$$

formally:
SMP.

Accessibility and recurrence

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Since $f_{a \to b} > 0$ by assumption, we must have $f_{b \to a} = 1$, so in particular, $b \to a$.

Accessibility and recurrence

- Let us now show that b is recurrent. $\Leftarrow \left\{ \begin{array}{l} a \text{ is RECURRENT} \\ a \to b \\ b \to a \end{array} \right\}.$

Accessibility and recurrence

will show $\mathbb{E}[N(b) \mid X_0 = b] = \infty$

pos. prob of going from a to b in exactly m steps.

- Let us now show that b is recurrent.
- Since $f_{a\to b} > 0$, $f_{b\to a} > 0$, we must have $p^m_{a,b} > 0$ and $p^{m'}_{b,a} > 0$ for some $m, m' > 0$.
- Note that for all $n \geq 0$

$$\mathbb{P}[X_{n+m+m'} = b \mid X_0 = b] \geq p^{m'}_{b,a} \cdot p^n_{a,a} \cdot p^m_{a,b}.$$

return to b in exactly $m' + n + m$ steps.

$$\begin{tikzcd} b \arrow[r, "m'"] \arrow[drr, "n+m+m'"'] & a \arrow[r, "n"] & a \arrow[d, "m"] \\ & & b \end{tikzcd}$$

Accessibility and recurrence

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$f_{a\to a} = \mathbb{P}(\tau_{\{a\},a} < \infty)$

- Summing this over all n, we have

$$\sum_{n=0}^{\infty} \mathbb{P}[X_{n+m+m'=b} \mid X_0 = b] \geq \overbrace{p^{m'}_{b,a}}^{>0} \cdot \overbrace{p^m_{a,b}}^{>0} \cdot \overbrace{\sum_{n=0}^{\infty} p^n(a,a)}^{=\infty}.$$

$$\mathbb{P}[X_n = a \mid X_0 = a] = \mathbb{E}[\mathbb{1}[X_n = a] \mid X_0 = a]$$

$$\sum_{n=0}^{\infty} \mathbb{P}[X_n = a \mid X_0 = a] = \mathbb{E}[N(a) \mid X_0 = a]$$

Accessibility and recurrence

- Let us now show that b is recurrent.
- Since $f_{a\to b} > 0$, $f_{b\to a} > 0$, we must have $p^{m}_{a,b} > 0$ and $p^{m'}_{b,a} > 0$ for some $m, m' > 0$.
- Note that for all $n \geq 0$

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- Summing this over all n, we have

$$\sum_{n=0}^{\infty} \mathbb{P}[X_{n+m+m'=b} \mid X_0 = b] \geq p^{m'}_{b,a} \cdot p^{m}_{a,b} \cdot \sum_{n=0}^{\infty} p^{n}(a,a).$$

- The RHS is infinite since a is recurrent and $p^{m}_{a,b} > 0, p^{m'}_{b,a} > 0$, which shows that the expected number of returns to b is infinite, and hence b is recurrent.

Decomposition of the state space

Let $S = S_1 \cup \cdots \cup S_k$ be the decomposition of the state space into communicating classes.

- By what we saw, for $i = 1, \ldots, k$, either all states in S_i are recurrent or all states in S_i are transient.

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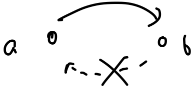
- By what we saw, for $i = 1, \ldots, k$, either all states in S_i are recurrent or all states in S_i are transient.
- We can say a bit more. Since

$$a \text{ is recurrent, and } a \to b \implies b \text{ is recurrent, and } b \to a,$$

it follows that

$$a \to b \text{ and } b \not\to a \implies a \text{ is transient.}$$



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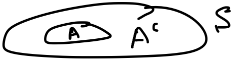
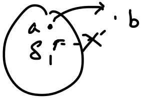
- We say that $A \subseteq S$ is **closed** if for all $a \in A$ and for all $b \in S \setminus A$, $a \not\to b$. In words, once we enter A, we do not exit A.



Decomposition of the state space

- Let $S = \overbrace{S_1' \cup \cdots \cup S_k'}^{\text{transient classes}} \cup \overbrace{S_1 \cup \ldots S_\ell}^{\text{Recurrent classes}}$ be a decomposition of the state space where S_i' are the transient communicating classes S_j are the recurrent communicating classes.
- It must be the case that $S_1, \ldots, S_\ell$ are closed.



$\Rightarrow$ a is transient

Decomposition of the state space

- Let $S = S_1' \cup \cdots \cup S_k' \cup S_1 \cup \ldots S_\ell$ be a decomposition of the state space where S_i' are the transient communicating classes S_j are the recurrent communicating classes.
- It must be the case that $S_1, \ldots, S_\ell$ are closed.
- Indeed, suppose S_i is not closed. Then, there must exist some $b \in S \setminus S_i$ such that $a \to b$. Since S_i is a maximal communicating class and $b \notin S_i$, we must have $b \not\to a$. But

$$a \to b \text{ and } b \not\to a \implies a \text{ is transient},$$

a contradiction.

Decomposition of the state space

- Also, it must be the case that $S'_1, \dots, S'_k$ are not closed.

equiva (*) if S'_i is finite, communicating & closed

$\Rightarrow$ S_i' is recurrent.

a recurrent $\Leftrightarrow$ $\mathbb{E}[N(a) \mid X_0 = a] = \infty$

Decomposition of the state space

- Also, it must be the case that $S_1', \ldots, S_k'$ are not closed.
- Indeed, suppose that S_i' is closed. Fix $a \in S_i'$ and note that

S_i' cannot leave.

$$\sum_{b \in S_i'} \mathbb{E}[N_{\delta_a}(b)] = \mathbb{E}\left[\sum_{b \in S_i'} N_{\delta_a}(b)\right] = \infty.$$

$N_{\delta_a}(b)$ = # of times you visit b.

Decomposition of the state space

- Also, it must be the case that $S_1', \ldots, S_k'$ are not closed.
- Indeed, suppose that S_i' is closed. Fix $a \in S_i'$ and note that

$$\sum_{b \in S_i'} \mathbb{E}[N_{\delta_a}(b)] = \mathbb{E}\left[\sum_{b \in S_i'} N_{\delta_a}(b)\right] = \infty.$$

- Therefore, there must exist some $b \in S_i'$ such that $\mathbb{E}[N_{\delta_a}(b)] = \infty$ and the same geometric random variable argument as before shows that $f_{b \to b} = 1$, which contradicts that b is transient.

Summary

Let $(X_n)_{n\geq 0}$ be a DTMC on a finite state space S. Then,

$$S = S_1' \cup \cdots \cup S_k' \cup C_1 \cup \ldots C_\ell, \quad \text{where}$$

- Each S_i', C_j is a communicating class.
- $S_1', \ldots, S_k', C_1, \ldots, C_\ell$ are disjoint.
- $S_1', \ldots, S_k'$ are not closed and all states in $S_1' \cup \cdots \cup S_k'$ are transient.
- Each C_i is closed and recurrent.
- $S_1' \cup \cdots \cup S_k' \neq S$. $\longrightarrow$ depends on S being finite.

Summary

Let $(X_n)_{n\geq 0}$ be a DTMC on a finite state space S. Then,

$$S = S_1' \cup \cdots \cup S_k' \cup C_1 \cup \ldots C_\ell, \quad \text{where}$$

(*) finite + closed set, then it has atleast one recurr. state.

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- $S_1', \ldots, S_k'$ are not closed and all states in $S_1' \cup \cdots \cup S_k'$ are transient.
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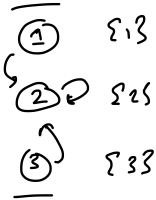

To see the last point, note that for any starting distribution μ_0,

$$\sum_{a\in S} \mathbb{E}[N_{\mu_0}(a)] = \mathbb{E}\left[\sum_{a\in S} N_{\mu_0}(a)\right] = \infty,$$

so that there must exist some $a \in S$ for which $\mathbb{E}[N_{\mu_0}(a)] = \infty$, and now the geometric random variable argument shows that $f_{a\to a} = 1$.