STATS 217: Introduction to Stochastic Processes I

Lecture 11

From last time

Let $(X_n)_{n\geq 0}$ be a DTMC on S with transition matrix P.

- $s \in S$ is recurrent if $f_{s\to s} = 1$, where $f_{s\to s} = \mathbb{P}[\tau_{\{s\},s} < \infty]$.
- We saw that s is recurrent if and only if

$$\mathbb{E}[N(s) \mid X_0 = s] = \infty,$$

  where $N(s)$ is the number of visits to s.
- While proving that a recurrent and $a \to b$ implies $b \to a$, we used that

$$\mathbb{P}[N(a) = \infty \mid X_0 = a] = 1.$$

  Note that this is stronger than saying that $\mathbb{E}[N(a) = \infty \mid X_0 = a] = \infty$.

From last time

- Why is this stronger statement true? It suffices to show that $\mathbb{P}[N(a) < \infty | X_0 = a] = 0$.
- By definition, $\{N(a) < \infty\} = \cup_{n \in \mathbb{Z}^{\geq 0}} \{N(a) = n\}$.
- We also know that for any $n \in \mathbb{Z}^{\geq 0}$

$$\mathbb{P}[N(a) = n \mid X_0 = a] = f^n_{a \to a} - f^{n+1}_{a \to a} = 0.$$

- Therefore,

$$\begin{aligned}
\mathbb{P}[N(a) < \infty \mid X_0 = 0] &= \sum_{n \in \mathbb{Z}^{\geq 0}} \mathbb{P}[N(a) = n \mid X_0 = a] \\
&= \sum_{n \in \mathbb{Z}^{\geq 0}} 0 \\
&= 0.
\end{aligned}$$

Exit distributions

- In the first lecture, we studied the Gambler's ruin: consider a gambler who bets on the outcome of fair coin tosses. What is the probability that she loses \$100 before winning \$200?
- We can study such questions more generally.
- For instance, generalizing our argument from Gambler's ruin shows the following.
- Let $(X_n)_{n \geq 0}$ be a DTMC on a finite state space S. Let $a \neq b \in S$ and let $C = S - \{a, b\}$. Let V_a be the first time (including 0) that a is visited and similarly for V_b. Suppose that $h(a) = 1$, $h(b) = 0$ and for all $x \in C$,

$$h(x) = \sum_{y \in S} p_{x,y} h(y).$$

If there exists some N such that $\mathbb{P}[\min\{V_a, V_b\} < N \mid X_0 = x] > 0$ for all $x \in C$, then

$$h(x) = \mathbb{P}[V_a < V_b \mid X_0 = x].$$

Exit distributions

- Let $T = \min\{V_a, V_b\}$.
- Since $\mathbb{P}[T < N \mid X_0 = x] > 0$ for all $x \in C$, the same argument as Problem 1 of HW1 shows that $\mathbb{P}[T < \infty] = 1$.
- The equation

$$h(x) = \sum_{y \in S} p_{x,y} h(y) \quad \forall x \in C.$$

can be rewritten as

$$h(x) = \mathbb{E}[h(X_1) \mid X_0 = x] \quad \forall x \in C.$$

- Iterating this, we have for all $x \in C$,

$$\begin{aligned} h(x) &= \mathbb{E}[h(X_T) \mid X_0 = x] \\ &= \mathbb{P}[X_T = a \mid X_0 = x] \\ &= \mathbb{P}[V_a < V_b \mid X_0 = x]. \end{aligned}$$

Example

Consider the following crude model of opinion dynamics.

- There is a population of N individuals, each with one of two opinions: A or B.
- Initially, $1 \leq x \leq N-1$ individuals have opinion A and $N-x$ individuals have opinion B.
- At each time step, the individuals update their opinion by sampling without replacement from the current opinions.
- This just means that if x people have opinion A today, then at the next time step, the probability that y people have opinion A is

$$p_{x,y} := \binom{N}{y} \left(\frac{x}{N}\right)^{y} \left(\frac{N-x}{N}\right)^{N-y} .$$

- What is the probability that everyone in the population eventually holds opinion A?

Example

- Let X_n denote the number of people with opinion A at time n.
- Then, X_n is a DTMC.
- We are interested in finding $\mathbb{P}[V_N < V_0 \mid X_0 = x]$.
- By the theorem, it suffices to find a function $h(x)$ with $h(N) = 1$, $h(0) = 0$ and for all $1 \leq x \leq N - 1$,

$$h(x) = \sum_{y \in S} p_{x,y} h(y).$$

- Since $p_{x,y} = \mathbb{P}[\mathrm{Binom}(N, x/N) = y]$, you can check easily that $h(x) = x/N$ is a valid choice.

A more general view

- Let $(X_n)_{n \geq 0}$ be a DTMC on a finite state space $S = \{1, \dots, N\}$ with transition matrix P.
- Suppose that all the recurrent states of S are absorbing.
- Without loss of generality, this means that there is some $r < N$ such that states $\{1, \dots, r\}$ are transient, states $\{r+1, \dots, N\}$ are recurrent, and $P_{x,x} = 1$ for all $x > r$.
- Therefore, the transition matrix P decomposes as

$$P = \begin{bmatrix} Q & R \\ 0 & I \end{bmatrix}$$

  where Q is an $r \times r$ matrix, R is an $r \times (N - r)$ matrix, and I is the $(N - r) \times (N - r)$ identity matrix.

A more general view

- Let T be the first time that the chain reaches one of the absorbing states. We know that $\mathbb{P}[T < \infty] = 1$.
- Our goal is to understand, for all $j > r$,

$$U_{i,j} = \mathbb{P}[X_T = j \mid X_0 = i].$$

- By definition, we must have $U_{j,j} = 1$ and $U_{i,j} = 0$ for all $i > r$, $i \neq j$.
- On the other hand, for any $i \leq r$, we have by first step analysis that

$$\begin{aligned} U_{i,j} &= P_{i,j} + \sum_{k \leq r} P_{i,k} U_{k,j} \\ &= R_{i,j} + \sum_{k \leq r} Q_{i,k} U_{k,j}, \end{aligned}$$

and by the same argument as before, a solution to these equations with the given boundary conditions gives $\mathbb{P}[X_T = j \mid X_0 = i]$.

Biased Gambler's ruin

- Let us return to the problem of the Gambler's ruin, except now, the bets are biased.
- Concretely, the gambler starts with $\$x$ and in each round, independently, wins \$1 with probability p and loses \$1 with probability q.
- She stops playing once she either reaches $\$N$ or \$0.
- We want to compute

$$h(x) = \mathbb{P}[V_N < V_0 \mid X_0 = x].$$

- As before, $h(N) = 1$, $h(0) = 0$ and for $1 \leq x \leq N - 1$,

$$h(x) = ph(x+1) + qh(x-1).$$

- Check that this is satisfied by

$$h(x) = \frac{\theta^x - 1}{\theta^N - 1}, \quad \theta = \frac{q}{p}.$$

Biased Gambler's ruin

- As an example, imagine that you are betting \$1 on each round of roulette, where there are 18 red, 18 black, and 2 green holes.
- In this case $p = 18/38$.
- So, for instance,

$$\begin{aligned}\mathbb{P}[V_{100} < V_{50} \mid X_0 = 50] &= \frac{(20/18)^{50} - 1}{(20/18)^{100} - 1} \\ &= 0.005128,\end{aligned}$$

which is almost 100 times less likely than when $p = 19/38$.