STATS 217: Introduction to Stochastic Processes I

Lecture 11

From last time

Let $(X_n)_{n\geq 0}$ be a DTMC on S with transition matrix P.

- $s \in S$ is recurrent if $f_{s\to s} = 1$, where $f_{s\to s} = \mathbb{P}[\tau_{\{s\},s} < \infty]$.
- We saw that s is recurrent if and only if

$$\mathbb{E}[N(s) \mid X_0 = s] = \infty,$$

where $N(s)$ is the number of visits to s.

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  where $N(s)$ is the number of visits to s.
- While proving that a recurrent and $a \to b$ implies $b \to a$, we used that

$$\mathbb{P}[N(a) = \infty \mid X_0 = a] = 1.$$

  Note that this is stronger than saying that $\mathbb{E}[N(a) = \infty \mid X_0 = a] = \infty$.

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- By definition, $\{N(a) < \infty\} = \cup_{n \in \mathbb{Z}^{\geq 0}} \{N(a) = n\}$.
- We also know that for any $n \in \mathbb{Z}^{\geq 0}$

$$\mathbb{P}[N(a) = n \mid X_0 = a] = f^n_{a \to a} - f^{n+1}_{a \to a} = 0.$$

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- Therefore,

$$\begin{aligned}\mathbb{P}[N(a) < \infty \mid X_0 = 0] &= \sum_{n \in \mathbb{Z}^{\geq 0}} \mathbb{P}[N(a) = n \mid X_0 = a] \\ &= \sum_{n \in \mathbb{Z}^{\geq 0}} 0 \\ &= 0.\end{aligned}$$

Exit distributions

- In the first lecture, we studied the Gambler's ruin: consider a gambler who bets on the outcome of fair coin tosses. What is the probability that she loses \$100 before winning \$200?
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- For instance, generalizing our argument from Gambler's ruin shows the following.
- Let $(X_n)_{n \geq 0}$ be a DTMC on a finite state space S. Let $a \neq b \in S$ and let $C = S - \{a, b\}$. Let V_a be the first time (including 0) that a is visited and similarly for V_b.

$S = [-100, -99, \ldots, 199, 200]$

$a = -100$

$b = 200$

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$$h(x) = \sum_{y \in S} p_{x,y} h(y).$$

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"first step analysis"

If there exists some N such that $\mathbb{P}[\min\{V_a, V_b\} < N \mid X_0 = x] > 0$ for all $x \in C$, then

"heads 300 times, then you win"

$$h(x) = \mathbb{P}[V_a < V_b \mid X_0 = x].$$

the prob. of hitting a before b starting at x.

Exit distributions

- Let $T = \min\{V_a, V_b\}$.
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- The equation

$$h(x) = \sum_{y \in S} p_{x,y} h(y) \quad \forall x \in C.$$

can be rewritten as

$$h(x) = \mathbb{E}[h(X_1) \mid X_0 = x] \quad \forall x \in C.$$

$$= \sum_y \mathbb{P}(X_1 = y \mid X_0 = x)\, h(y)$$

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$$h(x_1) = \begin{cases} x_1 \in \{a, b\} \\ \mathbb{E}[h(x_2) \mid x_1] \end{cases}$$

- Iterating this, we have for all $x \in C$,

$X_T \in \{a, b\}$

$h(a) = 1$, $h(b) = 0$.

$$\begin{aligned} h(x) &= \mathbb{E}[h(X_T) \mid X_0 = x] \\ &= \mathbb{P}[X_T = a \mid X_0 = x] \\ &= \mathbb{P}[V_a < V_b \mid X_0 = x]. \end{aligned}$$

$= \begin{cases} \mathbb{E}[h(x_T) \mid x_T = a, x_0 = x] \\ \cdot \mathbb{P}[x_T = a \mid x_0 = x] \end{cases}$

$+$ for b.

Example

Consider the following crude model of opinion dynamics.

- There is a population of N individuals, each with one of two opinions: A or B.
- Initially, $1 \leq x \leq N-1$ individuals have opinion A and $N-x$ individuals have opinion B.
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$$p_{x,y} := \binom{N}{y} \left(\frac{x}{N}\right)^{y} \left(\frac{N-x}{N}\right)^{N-y} = \mathbb{P}\left[\mathrm{Binom}\left(N, \frac{x}{N}\right) = y\right]$$

- What is the probability that everyone in the population eventually holds opinion A?

Example

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- By the theorem, it suffices to find a function $h(x)$ with $h(N) = 1$, $h(0) = 0$ and for all $1 \leq x \leq N - 1$,

$$h(x) = \sum_{y \in S} p_{x,y} h(y).$$

$$h(x) = \sum_{1}^{1} \underbrace{\mathbb{P}\left[\text{Binom}\left(N, \frac{x}{N}\right) = y\right]} h(y).$$

$$h(y) = y/N \qquad \begin{array}{l} h(N) = 1 \\ h(0) = 0 \end{array}$$

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- Since $p_{x,y} = \mathbb{P}[\text{Binom}(N, x/N) = y]$, you can check easily that $h(x) = x/N$ is a valid choice.

by thm, $h(x) = \mathbb{P}[V_N < V_0 \mid X_0 = x]$

A more general view

$$S = \underbrace{S_1' \cup \dots S_k'}^{\text{transient}} \cup \overset{\{c_1\}}{\overset{\shortparallel}{C_1}} \cup \dots \overset{\{c_\ell\}}{C_\ell}$$

- Let $(X_n)_{n\geq 0}$ be a DTMC on a finite state space $S = \{1, \dots, N\}$ with transition matrix P.
- Suppose that all the recurrent states of S are absorbing.
- Without loss of generality, this means that there is some $r < N$ such that states $\{1, \dots, r\}$ are transient, states $\{r + 1, \dots, N\}$ are recurrent, and $P_{x,x} = 1$ for all $x > r$.

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- Therefore, the transition matrix P decomposes as

prob. of going from transient → trans.

$$P = \begin{bmatrix} Q & R \\ 0 & I \end{bmatrix}$$

where Q is an $r \times r$ matrix, R is an $r \times (N - r)$ matrix, and I is the $(N - r) \times (N - r)$ identity matrix.

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- Let T be the first time that the chain reaches one of the absorbing states. We know that $\mathbb{P}[T < \infty] = 1$.
- Our goal is to understand, for all $j > r$,

$$U_{i,j} = \mathbb{P}[X_T = j \mid X_0 = i].$$

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absorbing states are $r+1, \ldots, N$

- By definition, we must have $U_{j,j} = 1$ and $U_{i,j} = 0$ for all $i > r$, $i \neq j$.

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- By definition, we must have $U_{j,j} = 1$ and $U_{i,j} = 0$ for all $i > r$, $i \neq j$.
- On the other hand, for any $i \leq r$, we have by first step analysis that

$$P = \begin{pmatrix} \boxed{Q} & \boxed{R} \\ 0 & I \end{pmatrix}$$

$$\begin{aligned} U_{i,j} &= P_{i,j} + \sum_{k \leq r} P_{i,k} U_{k,j} \quad (*) \\ &= R_{i,j} + \sum_{k \leq r} Q_{i,k} U_{k,j}, \end{aligned}$$

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and by the same argument as before, a solution to these equations with the given boundary conditions gives $\mathbb{P}[X_T = j \mid X_0 = i]$.

Biased Gambler's ruin

- Let us return to the problem of the Gambler's ruin, except now, the bets are biased.
- Concretely, the gambler starts with $\$x$ and in each round, independently, wins \$1 with probability p and loses \$1 with probability $q = 1-p$
- She stops playing once she either reaches $\$N$ or \$0.

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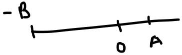
$$h(x) = ph(x+1) + qh(x-1). \implies p[h(x+1) - h(x)] = q[h(x) - h(x-1)]$$

$= p\,h(x) + q\,h(x)$

unbiased case



- Check that this is satisfied by

$$h(x) = \frac{\theta^x - 1}{\theta^N - 1}, \quad \theta = \frac{q}{p}.$$

A before $-B$ $= \frac{B}{A+B}$

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- In this case $p = 18/38$.

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- In this case $p = 18/38$.
- So, for instance,

$$\begin{aligned}\mathbb{P}[V_{100} < V_{50} \mid X_0 = 50] &= \frac{(20/18)^{50} - 1}{(20/18)^{100} - 1} \\ &= 0.005128,\end{aligned}$$

which is almost 100 times less likely than when $p = 19/38$.