STATS 217: Introduction to Stochastic Processes I

Lecture 12

Exit times

- Let $(X_n)_{n\geq 0}$ be a DTMC on a finite state space $S = \{1, \ldots, N\}$ with transition matrix P.
- Suppose that all the recurrent states of S are absorbing.
- Without loss of generality, this means that there is some $r < N$ such that states $\{1, \ldots, r\}$ are transient, states $\{r+1, \ldots, N\}$ are recurrent, and $P_{x,x} = 1$ for all $x > r$.
- Therefore, the transition matrix P decomposes as

$$P = \begin{bmatrix} Q & R \\ 0 & I \end{bmatrix}$$

  where Q is an $r \times r$ matrix, R is an $r \times (N-r)$ matrix, and I is the $(N-r) \times (N-r)$ identity matrix.

Exit times

- Let T be the first time that the chain reaches one of the absorbing states.
- We know that $\mathbb{P}[T < \infty] = 1$.
- Last time we studied the exit distribution starting from $i \in S$ i.e.,

$$U_{i,j} = \mathbb{P}[X_T = j \mid X_0 = i].$$

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- Today, we will study the **exit time** i.e. the random variable T itself.
- We already saw an example in the very first lecture when we discussed the expected time for a gambler to lose either \$$B$ or win \$$A$ when betting \$1 on the outcomes of independent, fair coin tosses.

Exit times

throughout : $1, \ldots, r$ are transient

$r+1, \ldots, N$ are absorbing

$P = \begin{pmatrix} Q & R \\ 0 & I \end{pmatrix}$ Q is $r \times r$ matrix.

- Fix $i \in \{1, \ldots, r\}$. What is $\mathbb{P}[T > t_0 \mid X_0 = i]$?
- Equivalently, this is the probability that $X_1, \ldots, X_{t_0} \in \{1, \ldots, r\}$ given that $X_0 = i$.

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  → + law of total prob.
- Using the Markov property, this probability is exactly

→ look at all possible values of $X_1, X_2, \dots, X_{t_0}$

$$\sum_{i_1, \dots, i_{t_0} \in \{1, \dots, r\}} P_{i,i_1} \overset{P_{i_1} P_{i_2}}{\cdots} P_{i_{t_0 - 1}, i_{t_0}}.$$

→ sum over all possible trajectories

Exit times

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- Using the Markov property, this probability is exactly

$$(*) \qquad \sum_{i_1, \ldots, i_{t_0} \in \{1, \ldots, r\}} P_{i,i_1} \ldots P_{i_{t_0-1}, i_{t_0}}.$$

(path counting).

$$P = \begin{pmatrix} \boxed{Q} & R \\ 0 & I \end{pmatrix}$$

- A more convenient way of writing this is as

$$\mathbb{P}[T > t_0 \mid X_0 = i] = \sum_{j=1}^{r} (Q^{t_0})_{i,j}.$$

Q is an $r \times r$ matrix

$$(Q^{t_0})_{ij} = \sum_{i_1 \ldots i_{t_0-1}} Q_{i i_1} Q_{i_1 i_2} \ldots Q_{i_{t_0-1} j}$$

Exit times

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$$g(i) = 1 + \sum_{j=1}^{r} P_{i,j}\, g(j).$$

(note that only summing over transient states.)

$\mathbb{E}[T \mid X_0 = i]$

take a step.

if hit $r+1, \dots, N$, then done

else hit $1, \dots, r$, in which case, restart from X_1.

Exit times

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last time : for exit distribution
$h(i) = \mathbb{E}[h(X_1) \mid X_0 = i]$

- Since for all $1 \leq i \leq r$,

$$g(i) = 1 + \mathbb{E}[g(X_1) \mid X_0 = i],$$

it follows from the same argument as in the last lecture that a solution to the above system of linear equations with the boundary conditions $g(r+1) = \cdots = g(N) = 0$ must satisfy $g(i) = \mathbb{E}[T \mid X_0 = i]$.

we know that $g(r+1) = \dots = g(N) = 0$.

$$P = \begin{pmatrix} Q & R \\ 0 & I \end{pmatrix}$$

- Another way of writing the previous result is as follows.
- Let $\vec{w} = (g(1), \dots, g(r)) \in \mathbb{R}^r$ and let $\vec{b} = (1, 1, \dots, 1) \in \mathbb{R}^r$. Then,

$$\vec{w} = \vec{b} + Q\vec{w} \quad \text{i.e.} \quad (I - Q)\vec{w} = \vec{b}.$$

ith component

$$g(i) = 1 + \sum_{j=1}^{r} P_{ij}\, g(j)$$

$$= 1 + \sum_{j=1}^{r} Q_{ij}\, g(j)$$

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provided that $(I - Q)^{-1}$ exists.

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$$\left\{ \frac{1}{1-x} = 1 + x + x^2 + \cdots \quad \text{for } |x| < 1. \right\}$$

- Therefore,

$$\vec{w} = (I - Q)^{-1}\vec{b},$$

"plug $x = Q$"

provided that $(I - Q)^{-1}$ exists.

- Since $(I - Q) \cdot (I + Q + Q^2 + Q^3 + \ldots) = I$, it follows that $I - Q$ is invertible if $I + Q + Q^2 + \ldots$ converges.

"neumann series expansion of $(I-Q)^{-1}$"

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- In our case, this convergence indeed holds since

$$0 \leq (Q^t)_{i,j} \leq \mathbb{P}[T > t \mid X_0 = i] \to 0 \text{ as } t \to \infty.$$

optional.

Biased Gambler's ruin

- Let us return to the problem of the Gambler's ruin, except now, the bets are biased.
- Concretely, the gambler starts with $\$x$ and in each round, independently, wins $1 with probability p and loses $1 with probability q.
- She stops playing once she either reaches $\$N$ or $0.

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- We want to compute
$$g(x) = \mathbb{E}[T \mid X_0 = x].$$
- We have, $g(N) = g(0) = 0$ and for $1 \leq x \leq N - 1$,
$$g(x) = 1 + pg(x+1) + qg(x-1).$$

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- Check that this is satisfied by

$$h(x) = \frac{x}{q-p} - \frac{N}{q-p} \cdot \frac{1-(q/p)^x}{1-(q/p)^N}.$$

$\left(q/p\right)^x$ also showed up in the exit distribution.

Biased Gambler's ruin

$\frac{q}{p} = 1.01$ (say)

$\frac{x}{q-p} - \frac{N}{(1.01)^N} \cdot C$

- As an example, consider the case when $p < q$.
- Then, as $N \to \infty$, we see that $h(x) \to \frac{x}{q-p}$.

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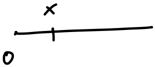

$N \to \infty$.

$$\lim_{N\to\infty} \mathbb{P}[V_N < V_0 \mid X_0 = x] = \lim_{N\to\infty} \frac{1-(q/p)^x}{1-(q/p)^N} = 0.$$

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- Intuition: As $N \to \infty$, we lose all our money with probability tending to 1. Moreover, since the expected loss per game is $(q-p)$ and since we start off with x, the expected number of steps is takes to lose all our money is $x/(q-p)$.

Patterns in coin tossing

TT : waiting time of 2.

HHHTHTT : waiting time of 6.

- You are tossing an unbiased coin repeatedly. What is the expected number of tosses to see the pattern TT? What is the expected number of tosses to see the pattern HT?

$H \to HT$, $H \to HH$

$T \to TT$, $T \to TH$

Patterns in coin tossing

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- The transition matrix is

$x_1\ x_2\ x_3\ x_4$

$$P = \begin{array}{c} \\ HH \\ HT \\ TH \\ TT \end{array} \begin{array}{c} \begin{array}{cccc} HH & HT & TH & TT \end{array} \\ \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix} \end{array}$$

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- You are tossing an unbiased coin repeatedly. What is the expected number of tosses to see the pattern TT? What is the expected number of tosses to see the pattern HT?
- The transition matrix is

$$Q$$

$$P = \begin{array}{c|cccc} & HH & HT & TH & TT \\ \hline HH & 1/2 & 1/2 & 0 & 0 \\ HT & 0 & 0 & 1/2 & 1/2 \\ TH & 1/2 & 1/2 & 0 & 0 \\ TT & 0 & 0 & \cancel{1/2} & \boxed{1} \end{array} \quad R. \quad = \begin{pmatrix} Q & R \\ 0 & I \end{pmatrix}$$

- In the case when we are waiting for TT, we modify the chain to make TT an absorbing state. In the case when we are waiting for HT, we modify the chain to make HT an absorbing state.

Patterns in coin tossing

- Let τ_{TT} denote the number of steps until we see TT.
- Let $\vec{w} = (w_{HH}, w_{HT}, w_{TH})$ where $w_{HH} = \mathbb{E}[\tau_{TT} \mid X_1 X_2 = HH]$ and so on.

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- Let $\vec{w} = (w_{HH}, w_{HT}, w_{TH})$ where $w_{HH} = \mathbb{E}[\tau_{TT} \mid X_1 X_2 = HH]$ and so on.
- Then, by our previous discussion, $(\underline{I} - Q)^{-1} \vec{b}$

$$\vec{w} = (I - Q)^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix},$$

starting state consists of 2 coin tosses.

where

$$Q = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

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$$\vec{w} = (I - Q)^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix},$$

where

$$Q = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

$$I - Q = \begin{pmatrix} 1/2 & -1/2 & 0 \\ 0 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{pmatrix}$$

- Solving this, we get

$$\vec{w} = \begin{pmatrix} 8 \\ 6 \\ 8 \end{pmatrix}$$

$\mathbb{E}[\tau_{TT} \mid X_1 X_2 = HH]$

$\mathbb{E}[\tau_{TT} \mid X_1 X_2 = HT]$

TH

Patterns in coin tossing

- Let τ_{TT} denote the number of steps until we see TT.
- Conditioning on the outcome of the first two tosses and using the law of total probability,

$$\mathbb{E}[\tau_{TT}] = \frac{\overset{HH}{8} + \overset{HT}{6} + \overset{TH}{8} + \overset{TT}{2}}{4} = 6.$$

Patterns in coin tossing

- Let τ_{TT} denote the number of steps until we see TT.
- Conditioning on the outcome of the first two tosses and using the law of total probability,

$$\mathbb{E}[\tau_{TT}] = \frac{8+6+8+2}{4} = 6.$$

- A similar computation shows that

$$\mathbb{E}[\tau_{HT}] = 4.$$

$$Q: \quad \begin{array}{c|ccc} & HH & TH & TT \\ \hline HH & & & \\ TH & & & \\ TT & & & \end{array}$$

$$(I - Q)^{-1}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \to \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$