STATS 217: Introduction to Stochastic Processes I

Lecture 14

Existence and uniqueness of stationary distributions

- The goal for today is to prove the following theorem from last time: let P be the transition matrix of an irreducible Markov chain on the state space S. Then, there exists a unique probability distribution π such that $\pi P = \pi$.
- First, we show uniqueness. Suppose that π and μ are probability distributions such that

$$\pi P = \pi, \qquad \mu P = \mu.$$

  We will show that $\pi = \mu$.
- Let x_* denote the state which minimizes the ratio $\pi(x)/\mu(x)$. Then, for all $y \in S$, we have

$$\frac{\pi(y)}{\mu(y)} \geq \frac{\pi(x_*)}{\mu(x_*)} =: \alpha.$$

Uniqueness of the stationary distribution

- Since π and μ are stationary distributions, we have for any $t \geq 1$ that

$$\begin{aligned} \pi(x_*) &= \sum_{y \in S} \pi(y) P^t_{y,x_*} \\ &\geq \sum_{y \in S} \alpha\mu(y) P^t_{y,x_*} \\ &\geq \alpha \sum_{y \in S} \mu(y) P^t_{y,x_*} \\ &= \alpha\mu(x_*) = \pi(x_*). \end{aligned}$$

- For this to hold, it must be the case that $\pi(y) = \alpha\mu(y)$ for every y such that $P^t_{y,x_*} > 0$.

Uniqueness of the stationary distribution

- Since P is irreducible, for every $y \in S$, there exists some t such that $P^t_{y,x_*} > 0$.
- Therefore, for all $y \in S$,

$$\pi(y) = \alpha\mu(y).$$

- But since both π and μ are probability distributions,

$$1 = \sum_{y \in S} \pi(y) = \alpha \sum_{y \in S} \mu(y) = \alpha,$$

so that $\alpha = 1$.

Existence of stationary distributions

- For $x, y \in S$, we define

$$\tau_{x \to y} = \min\{n \geq 1 : X_n = y\},$$

where $(X_n)_{n \geq 0}$ is a DTMC on S with transition matrix P starting from $X_0 = x$.

- Since P is irreducible, there exists some $r > 0$ such that for any $a, b \in S$, there exists some $j \leq r$ with $P^j_{a,b} > 0$.
- Then, by the geometric random variable argument we've seen many times,

$$\mathbb{E}[\tau_{x \to y} \mid X_0 = x] < \infty \qquad \forall x, y \in S.$$

Existence of stationary distributions

- We will explicitly construct of the stationary distribution.
- The idea is the following: imagine starting the chain at some $z \in S$, and breaking up the time into intervals based on returns to z. At each return to z, the chain starts afresh.
- Therefore, if we look at the expected fraction of time the chain spends in a state y between successive returns to z, then this should coincide with the long-term fraction of time spent by the chain in the state y.

Existence of stationary distributions

- This motivates the following definition.
- Fix $z \in S$ and let $(X_n)_{n \geq 0}$ be a DTMC with transition matrix P and $X_0 = z$. Define, for all $y \in S$,

$$\tilde{\pi}(y) = \mathbb{E}[\text{number of visits to } y \text{ before returning to } z]$$
$$= \sum_{t=0}^{\infty} \mathbb{P}[X_t = y, \tau_{z \to z} > t \mid X_0 = z].$$

- In particular, $\tilde{\pi}(z) = 1$.
- Also,

$$\sum_{y \in S} \tilde{\pi}(y) = \sum_{y \in S} \sum_{t=0}^{\infty} \mathbb{P}[X_t = y, \tau_{z \to z} > t \mid X_0 = z]$$
$$= \sum_{t=0}^{\infty} \mathbb{P}[\tau_{z \to z} > t \mid X_0 = z]$$
$$= \sum_{t=1}^{\infty} \mathbb{P}[\tau_{z \to z} \geq t \mid X_0 = z] = \mathbb{E}[\tau_{z \to z}].$$

Existence of stationary distributions

- Since $\mathbb{E}[\tau_{z\to z}] < \infty$, it follows that
$$\pi(y) := \frac{\tilde{\pi}(y)}{\mathbb{E}[\tau_{z\to z}]}$$
is a probability distribution on S.
- We will show that this is a stationary distribution for P. It suffices to show that
$$\tilde{\pi}P = \tilde{\pi}.$$

Existence of stationary distributions

- We will check this directly. Note that

$$\sum_{x \in S} \tilde{\pi}(x) P_{x,y} = \sum_{x \in S} \sum_{t=0}^{\infty} \mathbb{P}[X_t = x, \tau_{z \to z} > t \mid X_0 = z] \cdot P_{x,y}.$$

- Note that the event $\{\tau_{z \to z} > t\}$ is determined by $X_0, \ldots, X_t$. Therefore,

$$\begin{aligned} \mathbb{P}[X_t = x, X_{t+1} = y, \tau_{z \to z} > t \mid X_0 = z] &= \mathbb{P}[X_t = x, \tau_{z \to z} > t \mid X_0 = z] \\ &\quad \cdot \mathbb{P}[X_{t+1} = y \mid X_t = x, \tau_{z \to z} > t, X_0 = z] \\ &= \mathbb{P}[X_t = x, \tau_{z \to z} > t \mid X_0 = z] \cdot P_{x,y}. \end{aligned}$$

- Therefore, we can rewrite

$$\begin{aligned} \sum_{x \in S} \tilde{\pi}(x) P_{x,y} &= \sum_{x \in S} \sum_{t=0}^{\infty} \mathbb{P}[X_t = x, X_{t+1} = y, \tau_{z \to z} > t \mid X_0 = z] \\ &= \sum_{t=0}^{\infty} \mathbb{P}[X_{t+1} = y, \tau_{z \to z} > t \mid X_0 = z] \end{aligned}$$

Existence of stationary distributions

Continuing this, we have

$$\begin{aligned}\sum_{x\in S}\tilde{\pi}(x)P_{x,y} &= \sum_{t=0}^{\infty}\mathbb{P}[X_{t+1}=y,\tau_{z\to z}>t \mid X_0=z] \\ &= \sum_{t=0}^{\infty}\mathbb{P}[X_{t+1}=y,\tau_{z\to z}\geq t+1 \mid X_0=z] \\ &= \sum_{t=1}^{\infty}\mathbb{P}[X_{t}=y,\tau_{z\to z}\geq t \mid X_0=z] \\ &= \sum_{t=1}^{\infty}\mathbb{P}[X_{t}=y,\tau_{z\to z}> t \mid X_0=z] + \sum_{t=1}^{\infty}\mathbb{P}[X_{t}=y,\tau_{z\to z}= t \mid X_0=z]\end{aligned}$$

On the other hand

$$\tilde{\pi}(y) = \sum_{t=0}^{\infty}\mathbb{P}[X_t=y,\tau_{z\to z}>t \mid X_0=z].$$

Existence of stationary distributions

- Therefore,

$$\sum_{x \in S} \tilde{\pi}(x) P_{x,y} - \tilde{\pi}(y) = \left(\sum_{t=1}^{\infty} \mathbb{P}[X_t = y, \tau_{z \to z} = t \mid X_0 = z] \right) - \mathbb{P}[X_0 = y, \tau_{z \to z} > 0 \mid X_0 = z]$$
$$= \mathbb{1}[z = y] - \mathbb{P}[X_0 = y \mid X_0 = z]$$
$$= \mathbb{1}[z = y] - \mathbb{1}[z = y]$$
$$= 0.$$

- This shows that $\tilde{\pi}/\mathbb{E}[\tau_{z \to z}]$ is a stationary distribution, and by uniqueness, this is the only stationary distribution.
- In particular, for an irreducible Markov chain P on a finite state space S, the unique stationary distribution π is given by

$$\pi(z) = \frac{1}{\mathbb{E}[\tau_{z \to z}]} \qquad \forall z \in S$$