# STATS 217: Introduction to Stochastic Processes I

Lecture 14

# Existence and uniqueness of stationary distributions

- The goal for today is to prove the following theorem from last time: let $P$ be the transition matrix of an irreducible Markov chain on the state space $S$. Then, there exists a unique probability distribution $\pi$ such that $\pi P = \pi$.

# Existence and uniqueness of stationary distributions

- The goal for today is to prove the following theorem from last time: let $P$ be the transition matrix of an irreducible Markov chain on the state space $S$. Then, there exists a unique probability distribution $\pi$ such that $\pi P = \pi$.
- First, we show uniqueness. Suppose that $\pi$ and $\mu$ are probability distributions such that

  on $S$

  $$\pi P = \pi, \qquad \mu P = \mu.$$

  We will show that $\pi = \mu$.

# Existence and uniqueness of stationary distributions

- The goal for today is to prove the following theorem from last time: let $P$ be the transition matrix of an irreducible Markov chain on the state space $S$. Then, there exists a unique probability distribution $\pi$ such that $\pi P = \pi$.
- First, we show uniqueness. Suppose that $\pi$ and $\mu$ are probability distributions such that

$$\pi P = \pi, \qquad \mu P = \mu.$$

goal is to show that this is 1.

We will show that $\pi = \mu$.

- Let $x_*$ denote the state which minimizes the ratio $\pi(x)/\mu(x)$. Then, for all $y \in S$, we have

there could be multiple minima

$$\frac{\pi(y)}{\mu(y)} \geq \frac{\pi(x_*)}{\mu(x_*)} =: \alpha.$$

We will show that $\alpha = \underline{1}$.

# Uniqueness of the stationary distribution

$\pi P^t = \pi \quad \& \quad \mu P^t = \mu$

- Since $\pi$ and $\mu$ are stationary distributions, we have for any $t \geq 1$ that

$$\pi(x_*) = \sum_{y \in S} \pi(y) P^t_{y,x_*}$$

$$\geq \sum_{y \in S} \alpha \mu(y) P^t_{y,x_*}$$

$\pi(x_*) = (\pi P^t)(x^*) = \sum_y \pi(y) P^t_{y,x^*}$

$\pi(y) \geq \alpha \mu(y) \quad \forall y$

$\pi(x_*) = \alpha \mu(x_*) = \alpha \sum_y \mu(y) P^t_{y,x_*}$

$\alpha (\mu P^t)(x_*) = \alpha \mu(x_*) = \pi(x_*)$

# Uniqueness of the stationary distribution

- Since $\pi$ and $\mu$ are stationary distributions, we have for any $t \geq 1$ that

$$
\begin{aligned}
\pi(x_*) &= \sum_{y \in S} \pi(y) P^t_{y,x_*} \\
&\geq \sum_{y \in S} \alpha \mu(y) P^t_{y,x_*} \\
= \quad &\cancel{\geq} \alpha \sum_{y \in S} \mu(y) P^t_{y,x_*} \\
&= \alpha \mu(x_*) = \pi(x_*).
\end{aligned}
$$

$\geq$ must be $=$

which means that $\pi(y) = \alpha \mu(y)$ $\forall\, y$ s.t. $P^t_{y,x_*} > 0$.

# Uniqueness of the stationary distribution

- Since $\pi$ and $\mu$ are stationary distributions, we have for any $t \geq 1$ that

$$\pi(x_*) = \sum_{y \in S} \pi(y) P^t_{y,x_*}$$
$$\geq \sum_{y \in S} \alpha\mu(y) P^t_{y,x_*} \quad \Rightarrow \quad =$$
$$= \alpha \sum_{y \in S} \mu(y) P^t_{y,x_*} \quad =$$
$$= \alpha\mu(x_*) = \pi(x_*).$$

$\pi(y) \geq \alpha\mu(y)$

- $P^t_{y,x_*} = 0$, don't care
- $P^t_{y,x_*} > 0$.

- For this to hold, it must be the case that $\pi(y) = \alpha\mu(y)$ for every $y$ such that $P^t_{y,x_*} > 0$.

$\rightarrow$ $\pi(y) = \alpha\mu(y)$

$\times$ $\pi(y) > \alpha\mu(y)$ $\nearrow$ $\pi(y) P^t_{y,x_*} > \alpha\mu(y) P^t_{y,x_*}$

# Uniqueness of the stationary distribution

- Since $P$ is irreducible, for every $y \in S$, there exists some $t$ such that $P^t_{y,x_*} > 0$.
- Therefore, for all $y \in S$,

$$\pi(y) = \alpha\mu(y).$$

# Uniqueness of the stationary distribution

- Since $P$ is irreducible, for every $y \in S$, there exists some $t$ such that $P^t_{y,x_*} > 0$.
- Therefore, for all $y \in S$,

$$\pi(y) = \alpha\mu(y).$$

- But since both $\pi$ and $\mu$ are probability distributions,

$$1 = \sum_{y \in S} \pi(y) = \alpha \sum_{y \in S} \mu(y) = \alpha,$$

so that $\alpha = 1$.

another way

$\pi P = \pi$

$\pi (P - \underline{I}d) = 0$

# Existence of stationary distributions

- For $x, y \in S$, we define

$$\tau_{x \to y} = \min\{n \geq 1 : X_n = y\},$$

where $(X_n)_{n\geq 0}$ is a DTMC on $S$ with transition matrix $P$ starting from $X_0 = x$.

# Existence of stationary distributions

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  where $(X_n)_{n \geq 0}$ is a DTMC on $S$ with transition matrix $P$ starting from $X_0 = x$.
- Since $P$ is irreducible, there exists some $r > 0$ such that for any $a, b \in S$, there exists some $j \leq r$ with $P^j_{a,b} > 0$.

# Existence of stationary distributions

- For $x, y \in S$, we define

$$\tau_{x \to y} = \min\{n \geq 1 : X_n = y\},$$

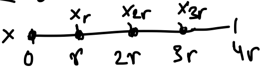
where $(X_n)_{n \geq 0}$ is a DTMC on $S$ with transition matrix $P$ starting from $X_0 = x$.

- Since $P$ is irreducible, there exists some $r > 0$ such that for any $a, b \in S$, there exists some $j \leq r$ with $P^j_{a,b} > 0$.
- Then, by the geometric random variable argument we've seen many times,

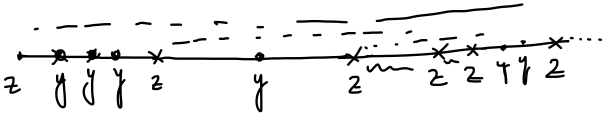
$$\mathbb{E}[\tau_{x \to y} \mid X_0 = x] < \infty \qquad \forall x, y \in S.$$

$a \xrightarrow{j_{a,b}} b.$

$r := \max_{a,b \in S} (j_{a,b}) < \infty$



# Existence of stationary distributions




- We will explicitly construct of the stationary distribution.
- The idea is the following: imagine starting the chain at some $z \in S$, and breaking up the time into intervals based on returns to $z$. At each return to $z$, the chain starts afresh.
- Therefore, if we look at the expected fraction of time the chain spends in a state $y$ between successive returns to $z$, then this should coincide with the long-term fraction of time spent by the chain in the state $y$.

$$\pi(y) = \left\{ \text{long term fraction of time spent in } y. \right\}$$

# Existence of stationary distributions

0 1 2 3 4 5 6
$z$ $z$ $\tau_{z\to z} = 6.$

- This motivates the following definition.
- Fix $z \in S$ and let $(X_n)_{n\geq 0}$ be a DTMC with transition matrix $P$ and $X_0 = z$. Define, for all $y \in S$,

include time 0, don't include $\tau_{z\to z}$

$$\tilde{\pi}(y) = \mathbb{E}[\text{number of visits to } y \text{ before returning to } z]$$

indicator random variables

$$= \sum_{t=0}^{\infty} \mathbb{P}[X_t = y, \tau_{z\to z} > t \mid X_0 = z].$$

count the visit to $y$ at time $t$ provided haven't returned to $z$ by time $t$.

- In particular, $\tilde{\pi}(z) = 1$.

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \mathbb{1}[X_t = y, \tau_{z\to z} > t] \mid X_0 = z\right]$$

# Existence of stationary distributions

- This motivates the following definition.
- Fix $z \in S$ and let $(X_n)_{n\geq 0}$ be a DTMC with transition matrix $P$ and $X_0 = z$. Define, for all $y \in S$,

$$\tilde{\pi}(y) = \mathbb{E}[\text{number of visits to } y \text{ before returning to } z]$$
$$= \sum_{t=0}^{\infty} \mathbb{P}[X_t = y, \tau_{z\to z} > t \mid X_0 = z].$$

- In particular, $\tilde{\pi}(z) = 1$. → we count the visit at time 0 but not the visit at time $\tau_{z\to z}$.
- Also,

$$\sum_{y\in S} \tilde{\pi}(y) = \sum_{y\in S}\sum_{t=0}^{\infty} \mathbb{P}[\underbrace{X_t = y}_{\text{can be ignored.}}, \tau_{z\to z} > t \mid X_0 = z]$$
$$= \sum_{t=0}^{\infty} \mathbb{P}[\tau_{z\to z} > t \mid X_0 = z]$$

$\sum_{y\in S} \mathbb{P}[\underbrace{X_t = y}, \ldots$

# Existence of stationary distributions

- This motivates the following definition.
- Fix $z \in S$ and let $(X_n)_{n \geq 0}$ be a DTMC with transition matrix $P$ and $X_0 = z$. Define, for all $y \in S$,

$$\begin{aligned}\tilde{\pi}(y) &= \mathbb{E}[\text{number of visits to } y \text{ before returning to } z] \\ &= \sum_{t=0}^{\infty} \mathbb{P}[X_t = y, \tau_{z\to z} > t \mid X_0 = z].\end{aligned}$$

- In particular, $\tilde{\pi}(z) = 1$.
- Also,

$$\begin{aligned}\sum_{y\in S} \tilde{\pi}(y) &= \sum_{y\in S} \sum_{t=0}^{\infty} \mathbb{P}[X_t = y, \tau_{z\to z} > t \mid X_0 = z] \\ &= \sum_{t=0}^{\infty} \mathbb{P}[\tau_{z\to z} > t \mid X_0 = z] \\ &= \sum_{t=1}^{\infty} \mathbb{P}[\tau_{z\to z} \geq t \mid X_0 = z] = \mathbb{E}[\tau_{z\to z}].\end{aligned}$$

Recall:

$X$ is nonneg int. valued

$\mathbb{E}[X] = \sum_{t=1}^{\infty} \mathbb{P}(X \geq t)$

# Existence of stationary distributions

Recall that

$$\mathbb{E}[\tau_{z\to z}] < \infty$$

by irreducibility of $P$.

- Since $\mathbb{E}[\tau_{z\to z}] < \infty$, it follows that

$$\pi(y) := \frac{\tilde{\pi}(y)}{\mathbb{E}[\tau_{z\to z}]} \geq 0$$

is a probability distribution on $S$.

$$\sum_{y \in S} \pi(y) = \frac{\sum \tilde{\pi}(y)}{\mathbb{E}[\tau_{z\to z}]} = \frac{\mathbb{E}[\tau_{z\to z}]}{\mathbb{E}[\tau_{z\to z}]} = 1.$$

•

# Existence of stationary distributions

- Since $\mathbb{E}[\tau_{z\to z}] < \infty$, it follows that

$$\pi(y) := \frac{\tilde{\pi}(y)}{\mathbb{E}[\tau_{z\to z}]}$$

  is a probability distribution on $S$.
- We will show that this is a stationary distribution for $P$. It suffices to show that

$$\tilde{\pi}P = \tilde{\pi}.$$

# Existence of stationary distributions

- We will check this directly. Note that

$$\sum_{x \in S} \tilde{\pi}(x) P_{x,y} = \sum_{x \in S} \sum_{t=0}^{\infty} \mathbb{P}[X_t = x, \tau_{z \to z} > t \mid X_0 = z] \cdot P_{x,y}.$$

wts. $= \tilde{\pi}(\cancel{x} y)$

$$= \sum_{t=0}^{\infty}{}' \mathbb{P}[X_t = y, \tau_{z \to z} > t \mid X_0 = z].$$

# Existence of stationary distributions

- We will check this directly. Note that

$$\sum_{x \in S} \tilde{\pi}(x) P_{x,y} = \sum_{x \in S} \sum_{t=0}^{\infty} \mathbb{P}[X_t = x, \tau_{z \to z} > t \mid X_0 = z] \cdot P_{x,y}.$$

- Note that the event $\{\tau_{z \to z} > t\}$ is determined by $X_0, \ldots, X_t$. Therefore,

$$\begin{aligned}\mathbb{P}[X_t = x, X_{t+1} = y, \tau_{z \to z} > t \mid X_0 = z] &= \mathbb{P}[X_t = x, \tau_{z \to z} > t \mid X_0 = z] \\ &\quad \cdot \mathbb{P}[X_{t+1} = y \mid X_t = x, \tau_{z \to z} > t, X_0 = z] \\ \text{markov prop.} \longrightarrow \quad &= \mathbb{P}[X_t = x, \tau_{z \to z} > t \mid X_0 = z] \cdot P_{x,y}.\end{aligned}$$

$$\sum_{x \in S} \cdots = \sum_{x \in S} \mathbb{P}[X_t = x, X_{t+1} = y, \tau_{z \to z} > t \mid X_0 = z].$$

die out because of the sum.

$\{\tau_{z \to z} > t\}$ depends only on $X_0 \ldots X_t$

# Existence of stationary distributions

- We will check this directly. Note that $\underbrace{\text{}}$ → double sum  wts. = single sum over t.

$$\sum_{x\in S} \tilde{\pi}(x) P_{x,y} = \sum_{x\in S}\sum_{t=0}^{\infty} \mathbb{P}[X_t = x, \underbrace{\tau_{z\to z} > t}_{\text{some info. about } X_0 \dots X_t} \mid X_0 = z] \cdot P_{x,y}.$$

- Note that the event $\{\tau_{z\to z} > t\}$ is determined by $X_0, \ldots, X_t$. Therefore,

$$\begin{aligned}\mathbb{P}[X_t = x, X_{t+1} = y, \tau_{z\to z} > t \mid X_0 = z] &= \mathbb{P}[X_t = x, \tau_{z\to z} > t \mid X_0 = z] \\ &\quad \cdot \mathbb{P}[X_{t+1} = y \mid X_t = x, \tau_{z\to z} > t, X_0 = z] \\ &= \mathbb{P}[X_t = x, \tau_{z\to z} > t \mid X_0 = z] \cdot P_{x,y}.\end{aligned}$$

- Therefore, we can rewrite

$$\begin{aligned}\sum_{x\in S} \tilde{\pi}(x) P_{x,y} &= \sum_{x\in S}\sum_{t=0}^{\infty} \mathbb{P}[X_t = x, X_{t+1} = y, \tau_{z\to z} > t \mid X_0 = z] \\ &= \sum_{t=0}^{\infty} \mathbb{P}[X_{t+1} = y, \tau_{z\to z} > t \mid X_0 = z]\end{aligned}$$

# Existence of stationary distributions $\tilde{\pi}(y)$

Continuing this, we have $= \boxed{\sum_{t=0}^{\infty}} \mathbb{P}[X_t = y, \boxed{\tau_{z\to z} > t} \mid X_0 = z]$

$$\sum_{x \in S} \tilde{\pi}(x) P_{x,y} = \sum_{t=0}^{\infty} \mathbb{P}[X_{t+1} = y, \tau_{z\to z} > t \mid X_0 = z]$$

$$= \sum_{t=0}^{\infty} \mathbb{P}[X_{t+1} = y, \tau_{z\to z} \geq t+1 \mid X_0 = z]$$

change of vars. $$= \sum_{t=1}^{\infty} \mathbb{P}[X_t = y, \tau_{z\to z} \geq t \mid X_0 = z]$$

$\to$ sum is going from 1 to $\infty$.

$\to$ $\tau_{z\to z} \geq t$

# Existence of stationary distributions

Continuing this, we have

$$
\begin{aligned}
\sum_{x \in S} \tilde{\pi}(x) P_{x,y} &= \sum_{t=0}^{\infty} \mathbb{P}[X_{t+1} = y, \tau_{z \to z} > t \mid X_0 = z] \\
&= \sum_{t=0}^{\infty} \mathbb{P}[X_{t+1} = y, \tau_{z \to z} \geq t+1 \mid X_0 = z] \\
&= \sum_{t=1}^{\infty} \mathbb{P}[X_t = y, \tau_{z \to z} \geq t \mid X_0 = z] \\
&= \sum_{t=1}^{\infty} \mathbb{P}[X_t = y, \tau_{z \to z} > t \mid X_0 = z] + \sum_{t=1}^{\infty} \mathbb{P}[X_t = y, \tau_{z \to z} = t \mid X_0 = z]
\end{aligned}
$$

only diff is 1 to ∞

# Existence of stationary distributions

Continuing this, we have

$$
\begin{aligned}
\sum_{x \in S} \tilde{\pi}(x) P_{x,y} &= \sum_{t=0}^{\infty} \mathbb{P}[X_{t+1} = y, \tau_{z \to z} > t \mid X_0 = z] \\
&= \sum_{t=0}^{\infty} \mathbb{P}[X_{t+1} = y, \tau_{z \to z} \geq t+1 \mid X_0 = z] \\
&= \sum_{t=1}^{\infty} \mathbb{P}[X_t = y, \tau_{z \to z} \geq t \mid X_0 = z] \\
&= \sum_{t=1}^{\infty} \mathbb{P}[X_t = y, \tau_{z \to z} > t \mid X_0 = z] + \sum_{t=1}^{\infty} \mathbb{P}[X_t = y, \tau_{z \to z} = t \mid X_0 = z]
\end{aligned}
$$

On the other hand

$$
\tilde{\pi}(y) = \sum_{t=0}^{\infty} \mathbb{P}[X_t = y, \tau_{z \to z} > t \mid X_0 = z].
$$

# Existence of stationary distributions

- Therefore,

$$\sum_{x \in S} \tilde{\pi}(x) P_{x,y} - \tilde{\pi}(y) = \left( \sum_{t=1}^{\infty} \mathbb{P}[X_t = y, \tau_{z \to z} = t \mid X_0 = z] \right) - \mathbb{P}[X_0 = y, \tau_{z \to z} > 0 \mid X_0 = z]$$

What is the prob. that when you return to z, you are at y

$=$

$\cancel{\mathbb{1}[z = y]}$

$=$

$\cancel{\mathbb{1}[z = y]}.$

# Existence of stationary distributions

- Therefore,

$$\begin{aligned}\sum_{x\in S}\tilde{\pi}(x)P_{x,y}-\tilde{\pi}(y) &= \left(\sum_{t=1}^{\infty}\mathbb{P}[X_t=y,\tau_{z\to z}=t \mid X_0=z]\right)-\mathbb{P}[X_0=y,\tau_{z\to z}>0 \mid X_0=z] \\ &= \mathbb{1}[z=y]-\mathbb{P}[X_0=y \mid X_0=z] \\ &= \mathbb{1}[z=y]-\mathbb{1}[z=y] \\ &= 0.\end{aligned}$$

# Existence of stationary distributions

- Therefore,

$$\begin{aligned}\sum_{x \in S} \tilde{\pi}(x) P_{x,y} - \tilde{\pi}(y) &= \left( \sum_{t=1}^{\infty} \mathbb{P}[X_t = y, \tau_{z \to z} = t \mid X_0 = z] \right) - \mathbb{P}[X_0 = y, \tau_{z \to z} > 0 \mid X_0 = z] \\ &= \mathbb{1}[z = y] - \mathbb{P}[X_0 = y \mid X_0 = z] \\ &= \mathbb{1}[z = y] - \mathbb{1}[z = y] \\ &= 0.\end{aligned}$$

- This shows that $\tilde{\pi}/\mathbb{E}[\tau_{z \to z}]$ is a stationary distribution, and by uniqueness, this is the only stationary distribution.

$$\pi(z) = \frac{\tilde{\pi}(z)}{\mathbb{E}[\tau_{z \to z}]} = \frac{1}{\mathbb{E}[\tau_{z \to z}]}.$$

# Existence of stationary distributions

- Therefore,

$$\sum_{x \in S} \tilde{\pi}(x) P_{x,y} - \tilde{\pi}(y) = \left( \sum_{t=1}^{\infty} \mathbb{P}[X_t = y, \tau_{z \to z} = t \mid X_0 = z] \right) - \mathbb{P}[X_0 = y, \tau_{z \to z} > 0 \mid X_0 = z]$$
$$= \mathbb{1}[z = y] - \mathbb{P}[X_0 = y \mid X_0 = z]$$
$$= \mathbb{1}[z = y] - \mathbb{1}[z = y]$$
$$= 0.$$

"standard way of proving exis + uniq."
Perron-frobenius thm.

- This shows that $\tilde{\pi}/\mathbb{E}[\tau_{z \to z}]$ is a stationary distribution, and by uniqueness, this is the only stationary distribution.
- In particular, for an irreducible Markov chain $P$ on a finite state space $S$, the unique stationary distribution $\pi$ is given by

$$\pi(z) = \frac{1}{\mathbb{E}[\tau_{z \to z}]} \qquad \forall z \in S$$