STATS 217: Introduction to Stochastic Processes I

Lecture 19

Convergence theorem

Today, we will prove the convergence theorem for irreducible, aperiodic, finite-state Markov chains.

Let $(X_n)_{n \geq 0}$ be a DTMC on S with transition matrix P. Suppose that P is irreducible and aperiodic with unique stationary distribution π. There exists some $\epsilon > 0$ (depending on P) such that

$$\max_{x \in S} \mathsf{TV}(X_n \mid X_0 = x, \pi) \leq (1 - \epsilon)^n.$$

Key quantities

- For any $x \in S$, let
$$\Delta_x(n) = \mathrm{TV}(X_n \mid X_0 = x, \pi).$$
- Let
$$\Delta(n) = \max_{x \in S} \Delta_x(n).$$
- Therefore, our goal is to show that there exists some $\epsilon > 0$ such that
$$\Delta(n) \leq (1 - \epsilon)^n.$$

- On the homework, you will show that $\Delta(n+1) \leq \Delta(n)$ for all integers $n \geq 0$.
- Therefore, it suffices to show that there is some integer $r_0 \geq 1$ and some $\epsilon > 0$ such that
$$\Delta(r_0 n) \leq (1 - \epsilon)^n.$$

Key quantities

- It will be a bit more convenient to work with the following quantities: for any $x, y \in S$, let
$$D_{x,y}(n) = \mathrm{TV}(X_n \mid X_0 = x, X_n \mid X_0 = y).$$
- Let
$$D(n) = \max_{x,y \in S} D_{x,y}(n).$$
- $\Delta(n) \leq D(n)$ for all integers n. Why?
- It suffices to show that for any $x \in S$ and any $A \subseteq \Omega$,
$$\mathbb{P}[X_n \in A \mid X_0 = x] - \pi(A) \leq D(n).$$

Key quantities

We have

$$\begin{aligned}
\mathbb{P}[X_n \in A \mid X_0 = x] - \pi(A) &= P^n(x, A) - \sum_{y \in S} \pi(y) P^n(y, A) \\
&= \sum_{y \in S} \pi(y) P^n(x, A) - \sum_{y \in S} \pi(y) P^n(y, A) \\
&= \sum_{y \in S} \pi(y) [P^n(x, A) - P^n(y, A)] \\
&\leq \max_{y \in S} |P^n(x, A) - P^n(y, A)| \\
&\leq \max_{y \in S} D_{x,y}(n) \\
&\leq D(n).
\end{aligned}$$

Overview

- We want to show that there is some integer $r_0 \geq 1$ and some $\epsilon > 0$ such that
$$\Delta(r_0 n) \leq (1-\epsilon)^n.$$
- Since $\Delta(n) \leq D(n)$ for all integers $n \geq 0$, it suffices to show that there is some integer $r_0 \geq 1$ and some $\epsilon > 0$ such that
$$D(r_0 n) \leq (1-\epsilon)^n.$$
- For this, we will first show that D is sub-multiplicative i.e., for any integers $s, t \geq 0$,
$$D(s+t) \leq D(s)D(t).$$
- This implies that for any integer $r \geq 1$, $D(nr) \leq D(r)^n$.
- Finally, using the irreducibility and aperiodicity of P, we will show that there exists an integer $r_0 \geq 1$ such that $D(r_0) < 1$.

Sub-multiplicativity of D

- Let us prove the key sub-multiplicativity property

$$D(t+s) \leq D(t)D(s).$$

- The left hand side is

$$\max_{x,y \in S} \mathrm{TV}(X_{t+s} \mid X_0 = x, X_{t+s} \mid X_0 = y).$$

- For now, fix $x, y \in S$. Later, we will take the maximum.
- We will bound the left hand side by constructing a coupling $(\widehat{X}_{t+s}, \widehat{Y}_{t+s})$ of $X_{t+s} \mid X_0 = x$ and $X_{t+s} \mid X_0 = y$.

Constructing a coupling

Here is our coupling:

- First, use the coupling lemma to find a coupling $(\widehat{X}_t, \widehat{Y}_t)$ of the distributions $X_t \mid X_0 = x$ and $X_t \mid X_0 = y$ such that

$$\mathbb{P}[\widehat{X}_t \neq \widehat{Y}_t] = \mathrm{TV}(X_t \mid X_0 = x, X_t \mid X_0 = y) = D_{x,y}(t).$$

- If $\widehat{X}_t = \widehat{Y}_t$, then set $\widehat{X}_{t+s} = \widehat{Y}_{t+s}$.
- Else, if $x' = \widehat{X}_t \neq \widehat{Y}_t = y'$, use the coupling lemma to find a coupling $(\widehat{U}_s, \widehat{W}_s)$ of the distributions $X_{t+s} \mid X_t = x'$ and $X_{t+s} \mid X_t = y'$ such that

$$\mathbb{P}[\widehat{U}_s \neq \widehat{W}_s] = \mathrm{TV}(X_{t+s} \mid X_t = x', X_{t+s} \mid X_t = y') = D_{x',y'}(s) \leq D(s).$$

  Here, the second equality uses the Markov property. Then, set

$$(\widehat{X}_{t+s}, \widehat{Y}_{t+s}) = (\widehat{U}_s, \widehat{W}_s).$$

Analysis of the coupling

- By construction, it is clear that $\widehat{X}_{t+s} \sim X_{t+s} \mid X_0 = x$ and $\widehat{Y}_{t+s} \sim X_{t+s} \mid X_0 = y$.
- Therefore, by the coupling lemma,

$$\begin{aligned} D_{x,y}(t+s) &\leq \mathbb{P}[\widehat{X}_{t+s} \neq \widehat{Y}_{t+s}] \\ &= \mathbb{P}[\widehat{X}_t \neq \widehat{Y}_t, \widehat{U}_s \neq \widehat{W}_s] \\ &= \mathbb{P}[\widehat{U}_s \neq \widehat{W}_s \mid \widehat{X}_t \neq \widehat{Y}_t]\mathbb{P}[\widehat{X}_t \neq \widehat{Y}_t] \\ &\leq D(s)D_{x,y}(t). \end{aligned}$$

- Taking the maximum over all $x, y \in S$, we get that

$$D(t+s) \leq D(s)D(t).$$

Bounding $D(r)$

- We claim that $D(r_0) < 1$.
- On the homework, you will show that if P is irreducible and aperiodic, then there exists some r_0 such that $P^{r_0}_{x,y} > 0$ for all $x, y \in S$.
- In particular, for any $x \in S$ and for any $A \subseteq S$, $A \neq \emptyset$, we have $P^{r_0}(x, A) > 0$.
- For such an r_0, for any $x, y \in S$ and for any $A \subseteq S$, $A \neq \emptyset$, we have

$$|P^{r_0}(x, A) - P^{r_0}(y, A)| \leq |1 - \min\{P^{r_0}(x, A), P^{r_0}(y, A)\}| < 1.$$

- Taking the maximum over all $A \neq \emptyset$ shows that

$$D_{x,y}(r_0) = \mathrm{TV}(X_{r_0} \mid X_0 = x, X_{r_0} \mid X_0 = y) < 1.$$

- Finally, taking the maximum over all $x, y \in S$, we have $D(r_0) < 1$.