

STATS 217: Introduction to Stochastic Processes I

Lecture 21

today + next time: mixing times

next week: CTMC

last week: CTMC + martingales

advertisement: next quarter
Stats 318

Modern Markov Chains

Stationary times

- Last time, we used the following description of the lazy random walk on the hypercube: suppose the current state is x and the current time is t . We choose a coordinate $i_{t+1} \in \{1, \dots, n\}$ uniformly at random and an unbiased bit $b_{t+1} \in \{0, 1\}$, also uniformly at random, and independently of the coordinate i_{t+1} . Then, we set the value of coordinate i_{t+1} to b_{t+1} and keep all other coordinates unchanged.

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- Let τ_{refresh} be the first time that all coordinates have been chosen to be updated.
- Note that τ_{refresh} is a stopping time with respect to the collection of random tuples $\{(i_k, b_k)\}_{k \geq 1}$. (Why?)

Recall this means that the event
 $\tau_{\text{refresh}} = n$ is determined by
 $i_1, b_1, \dots, i_n, b_n$.
this is not a

stopping time wrt X_t

Stationary times

$$n=3 \quad 0 \ 0 \ 0$$

$$i_1 = 1 \quad b_1 = 0$$

$$i_2 = 2 \quad b_2 = 0$$

$$i_3 = 3 \quad b_3 = 0$$

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- Let τ_{refresh} be the first time that all coordinates have been chosen to be updated.
- Note that τ_{refresh} is a stopping time with respect to the collection of random tuples $\{(i_k, b_k)\}_{k \geq 1}$. (Why?)
- Moreover, note that $X_{\tau_{\text{refresh}}}$ is distributed uniformly on $\{0, 1\}^n$. (Why?)

τ_{refresh} is random
 $X_{\tau_{\text{refresh}}}$

* τ_{refresh} depends only on i_1, i_2, i_3, \dots

$$(X_{\tau_{\text{refresh}}})_j = \bigwedge_{i \leq j} b_i \text{ where } i \text{ is max } i$$

Strong stationary times

before
 τ_{refresh} s.t.
coord chosen
at $i = j$ }

- τ_{refresh} is an example of a **stationary time**.
- By this, we mean a stopping time τ (with respect to the initial state x of the chain as well as any auxiliary randomness) which satisfies

$$\mathbb{P}[X_\tau = y \mid X_0 = x] = \pi(y) \quad \forall y \in S,$$

where π is the stationary distribution of the chain.

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- Let $x \in S$ and consider the chain with initial state $X_0 = x$. A stationary time τ for which τ and X_τ are independent i.e. for all integers t and $y \in S$

$$\mathbb{P}[\tau = t, X_\tau = y \mid X_0 = x] = \mathbb{P}[\tau = t \mid X_0 = x] \cdot \pi(y)$$

is called a **strong stationary time** for the starting state x .

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- τ_{refresh} is an example of a strong stationary time for any starting state x . (Why?)

Strong stationary times

- A strong stationary time for the starting state x is a stopping time (with respect to the chain and auxiliary randomness) satisfying

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\Downarrow + law of total probability

- The property that will be useful to us is

$$\left[\mathbb{P}[\tau \leq t, X_t = y \mid X_0 = x] = \mathbb{P}[\tau \leq t \mid X_0 = x] \cdot \underbrace{\pi(y)} \right]$$

which can be proved using the law of total probability.

Strong stationary times

Indeed,

$$\mathbb{P}[\tau \leq t, X_t = y \mid X_0 = x] = \sum_{s \leq t} \sum_{z \in S} \mathbb{P}[\tau = s, X_t = y, X_s = z \mid X_0 = x]$$

Strong stationary times

Indeed,

$$\begin{aligned}\mathbb{P}[\tau \leq t, X_t = y \mid X_0 = x] &= \sum_{s \leq t} \sum_{z \in S} \mathbb{P}[\tau = s, X_t = y, X_s = z \mid X_0 = x] \\ &= \sum_{s \leq t} \sum_{z \in S} \mathbb{P}[\tau = s, X_s = z \mid X_0 = x] \cdot P_{z,y}\end{aligned}$$

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Bounding the mixing time using strong stationary times

- Strong stationary times are also a useful tool for bounding the mixing time. This is captured by the following:
- Suppose that τ is a strong stationary time for the starting state x . Then,

$$\text{TV}(X_t | X_0 = x, \pi) \leq \mathbb{P}[\tau > t | X_0 = x]. \quad * \text{ not as autological.}$$

Recall that last time :

$$\left\{ \begin{aligned} D_{xy}(t) &= \text{TV}(X_t | X_0 = x, X_t | X_0 = y) \\ &\leq \mathbb{P}[\tau_{\text{couple}} > t] \end{aligned} \right\}$$

τ_{couple} = coalescence time for (\hat{X}_t, \hat{Y}_t) .

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- Note that

$$\begin{aligned} \text{TV}(X_t | X_0 = x, \pi) &= \sum_{y: \pi(y) > P_{x,y}^t} [\pi(y) - P_{x,y}^t] \\ &\leq \frac{1}{2} \sum_y |\pi(y) - P_{x,y}^t| \\ &= \frac{1}{2} (|p - q| + \frac{1}{2} |1 - p - 1 + q|) \\ &= \frac{1}{2} |p - q| \end{aligned}$$

$\text{TV}(\text{Ber}(p), \text{Ber}(q))$

Bounding the mixing time using strong stationary times

- Strong stationary times are also a useful tool for bounding the mixing time. This is captured by the following:

- Suppose that τ is a strong stationary time for the starting state x . Then,

last time:

$$\tau_{\text{mix}}(\varepsilon) \leq n \log n + n \log(\varepsilon^{-1}).$$

e.g. $\{0, 1\}^n$
 $\tau = \tau_{\text{refresh}}$

- Note that

$$\begin{aligned} \text{TV}(X_t \mid X_0 = x, \pi) &= \sum_{y: \pi(y) > P_{x,y}^t} [\pi(y) - P_{x,y}^t] \\ &= \sum_{y: \pi(y) > P_{x,y}^t} \pi(y) \left(\underbrace{1 - \frac{P_{x,y}^t}{\pi(y)}}_{\text{nonnegative}} \right) \\ \sum \pi(y) &\leq 1 \end{aligned}$$

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separation distance



Bounding the mixing time using strong stationary times

- Hence, it will suffice to show that for all $y \in S$,

$$1 - \frac{P_{x,y}^t}{\pi(y)} \leq \mathbb{P}[\tau > t \mid X_0 = x].$$

$$\begin{aligned} \Rightarrow \text{TV}(P^t(x, \cdot), \pi) &\leq \max_{y \in S} \left(1 - \frac{P_{x,y}^t}{\pi(y)} \right) \\ &\leq \mathbb{P}[\tau > t \mid X_0 = x]. \end{aligned}$$

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- We have

$$\begin{aligned} 1 - \frac{P_{x,y}^t}{\pi(y)} &= 1 - \frac{\mathbb{P}[X_t = y \mid X_0 = x]}{\pi(y)} \\ &\leq 1 - \frac{\mathbb{P}[X_t = y, \tau \leq t \mid X_0 = x]}{\pi(y)} \\ &= 1 - \frac{\cancel{\pi(y)} \mathbb{P}[\tau \leq t \mid X_0 = x]}{\cancel{\pi(y)}} \\ &= \mathbb{P}[\tau > t \mid X_0 = x]. \end{aligned}$$

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Bounding the mixing time using strong stationary times

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* strong stationary times

$$\left. \begin{aligned} & \text{TV}(P^t(x, \cdot), \pi) \\ & \leq \mathbb{P}[\tau > t \mid X_0 = x] \end{aligned} \right\}$$

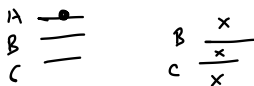
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* in examples:

we want to find a not wasteful τ .

Example: top-to-random shuffle



- Consider a deck of n cards.
- At each step of the top-to-random shuffle, we take the top card, and place it in any of the n available positions.

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- On the homework, you showed that this is an aperiodic and irreducible chain with the unique stationary distribution π given by the uniform distribution on all possible permutations of the n cards.

$$\begin{aligned} \# \text{ of permutation of } \{1, \dots, n\} \\ &= n! \\ &\sim \dots \left(\frac{n}{e}\right)^n \end{aligned}$$

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- At each step of the top-to-random shuffle, we take the top card, and place it in any of the n available positions.
- On the homework, you showed that this is an aperiodic and irreducible chain with the unique stationary distribution π given by the uniform distribution on all possible permutations of the n cards.
- Now, we will bound the mixing time by constructing a suitable strong stationary time.

Example: top-to-random shuffle

- Let τ_{top} be first time when the original bottom card has moved to the top.

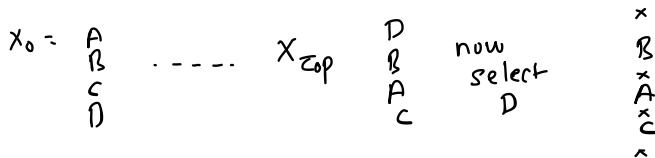
A —
B —
C —

• is τ_{top} a stationary time?

$$\mathbb{P}[X_{\tau_{\text{top}}} = y \mid X_0 = \begin{matrix} A \\ B \\ C \end{matrix}] = \pi(y).$$

Example: top-to-random shuffle

- Let τ_{top} be first time when the original bottom card has moved to the top.
- Then, $\tau = \tau_{\text{top}} + 1$ is a strong stationary time.



Example: top-to-random shuffle

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- Let τ_{top} be first time when the original bottom card has moved to the top.
- Then, $\tau = \tau_{\text{top}} + 1$ is a strong stationary time.
- Clearly, this is a stopping time.
- The key to showing that it is a strong stationary time is the following observation, which can be proved by induction: let τ_k denote the first time when there are k cards under the original bottom card. Then, at X_{τ_k} , all $k!$ permutations of these cards are equally likely. (Why?)

e.g.

$X_0 =$

A
B
C
D

- e.g. X_1

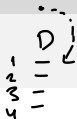
B
C
D
A

e.g.

C
D
A
X

we know
that B
goes below
D

Example: top-to-random shuffle



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- The key to showing that it is a strong stationary time is the following observation, which can be proved by induction: let τ_k denote the first time when there are k cards under the original bottom card. Then, at X_{τ_k} , all $k!$ permutations of these cards are equally likely. (Why?)
- Therefore, by the relationship between strong stationary times and total variation distance,

$$\text{TV}(X_t \mid X_0 = x, \pi) \leq \mathbb{P}[\tau > t \mid X_0 = x].$$

• A
• B
• C
• D.

prob [A goes below D] = $1/4$

D
A

Example: top-to-random shuffle

$$\mathcal{D} \begin{array}{c} \text{---} \\ 1 \\ 2 \\ 3 \end{array}$$

$\tau = \tau_{\text{top}} + 1$
first time when all
cards are below
original bottom card.

- Note that, when there are k cards under the bottom card, the probability that the top card goes under it is $(k+1)/n$.
- Therefore, we see that

$$\tau \sim \text{Geom}(1/n) + \text{Geom}(2/n) + \dots + \text{Geom}(1),$$

where the geometric random variables on the right hand side are independent. This is the same as the distribution of the coupon collector's time.

- Therefore, as for the lazy random walk on the hypercube, we get that

$$\tau_{\text{mix}}(\varepsilon) \leq n \log n + n \log(\varepsilon^{-1}).$$

$$\begin{aligned} n &= 52 \\ \varepsilon &= 0.01 \end{aligned}$$

~ 193.2

next time!
Riffle
shuffle
 $\tau_{\text{mix}}(1/4) \sim (\log)^n$