STATS 217: Introduction to Stochastic Processes I

Lecture 26

Martingales

- Let $X_1, X_2, \dots$ be a collection of random variables.
- We say that the sequence of random variables $M_0, M_1, \dots$ is a **martingale** with respect to $X_1, X_2, \dots$ if
  - $\mathbb{E}[|M_n|] < \infty$ for all $n \geq 0$,
  - for all $n \geq 1$, there exists a function $f_n : \mathbb{R}^n \to \mathbb{R}$ such that

$$M_n = f_n(X_1, \dots, X_n),$$

  - $\mathbb{E}[M_n \mid X_1, X_2, \dots, X_{n-1}] = M_{n-1}$. Explicitly, for any $x_1, \dots, x_{n-1}$,

$$\mathbb{E}[M_n \mid X_1 = x_1, \dots, X_{n-1} = x_{n-1}] = f_{n-1}(x_1, \dots, x_{n-1}).$$

Example

- $X_1, X_2, \ldots$ are independent random variables with $\mathbb{E}[X_i] = 0$ for all $i \geq 1$.
- Let $M_0 = 0$ and for $n \geq 1$,

$$M_n = X_1 + \cdots + X_n.$$

- Then, $M_0, M_1, \ldots$ is a martingale with respect to $X_1, X_2, \ldots$.
- This generalizes the one-dimensional simple, symmetric random walk.

Example

- $X_1, X_2, \ldots$ are independent random variables with $\mathbb{E}[X_i] = 0$ and $\mathrm{Var}(X_i) = \sigma^2$ for all $n \geq 1$.
- Then, $M_0 = 0$ and for $n \geq 1$,

$$M_n = (X_1 + \cdots + X_n)^2 - n\sigma^2$$

  is a martingale with respect to $X_1, X_2, \ldots$.
- To verify the martingale property, note that

$$\begin{aligned}
\mathbb{E}[M_n - M_{n-1} \mid X_1, \ldots, X_{n-1}] &= \mathbb{E}[(X_n + S_{n-1})^2 - S_{n-1}^2 - \sigma^2 \mid X_1, \ldots, X_{n-1}] \\
&= \mathbb{E}[X_n^2 + 2X_n S_{n-1} - \sigma^2 \mid X_1, \ldots, X_{n-1}] \\
&= \mathbb{E}[2X_n S_{n-1} \mid X_1, \ldots, X_{n-1}] \\
&= 2S_{n-1}\mathbb{E}[X_n \mid X_1, \ldots, X_{n-1}] \\
&= 0
\end{aligned}$$

Example

- $X_1, X_2, \ldots$ are independent random variables with $X_i \geq 0$ and $\mathbb{E}[X_i] = 1$ for all $i \geq 1$.
- Then, $M_0 = 1$ and for $n \geq 1$,

$$M_n = M_0 \cdot X_1 \cdots X_n$$

is a martingale with respect to $X_1, \ldots, X_n$.

Example

- Let $Y_1, Y_2, \ldots$ be i.i.d. random variables with moment generating function

$$\phi(\lambda) := \mathbb{E}[e^{\lambda Y_i}] < \infty$$

- Let $X_i = e^{\lambda Y_i}/\phi(\lambda)$. Then, $X_1, X_2, \ldots$ are independent random variables with $\mathbb{E}[X_i] = 1$.
- Therefore, $M_0 = 1$ and for $n \geq 1$,

$$M_n = M_0 \cdot X_1 \cdots X_n = e^{\lambda(Y_1 + \cdots + Y_n)}/\phi(\lambda)^n$$

is a martingale with respect to $Y_1, Y_2, \ldots$.

Example

- Consider a branching process $(Z_n)_{n\geq 0}$ with $Z_0 = 1$ and common offspring distribution ξ with $\mathbb{E}[\xi] = \mu \in (0, \infty)$.
- Recall this means that

$$Z_n = \sum_{i=1}^{Z_{n-1}} \xi_i,$$

  where $\xi_1, \xi_2, \ldots$ are i.i.d. copies of ξ.
- The sequence $M_0 = 1$ and for $n \geq 1$,

$$M_n = \frac{Z_n}{\mu^n}$$

  is a martingale with respect to $M_1, M_2, \ldots$.

Submartingales and supermartingales

- A **supermartingale** is defined similarly to a martingale, except now we weaken the martingale condition to

$$\mathbb{E}[M_n \mid X_1, \ldots, X_{n-1}] \leq M_{n-1}.$$

- Thinking of X_i as being the outcome of the i^{th} round of the gambling game, and M_n as being the wealth of the gambler after n rounds of the game, supermartingales are games that are unfavorable to the gambler.
- On the other hand, **submartingales** are favorable to the gambler i.e., they satisfy

$$\mathbb{E}[M_n \mid X_1, \ldots, X_{n-1}] \geq M_{n-1}.$$

Martingale betting strategy

- Consider a gambling game based on successive outcomes of a fair coin toss.
- You adopt the following strategy: if you win a round, then in the next round, you bet \$1; if you lose a round, then in the next round, you double your bet.
- So, for instance, if you lose in the first three rounds, and win in the fourth round, your sequence of bets is \$1, \$2, \$4, \$8, and your net winnings are

$$-\$1 - \$2 - \$4 + \$8 = 1.$$

- More generally, if you lose the first k rounds and win the $k + 1^{st}$ round, your net winnings are

$$-\$(1 + \cdots + 2^{k-1}) + \$2^k = \$1.$$

- Moreover, in an infinite sequence of fair coin tosses, you will win with probability 1.

Martingale betting strategy

- Let's take a look at this game for a fixed number of rounds, say 3 rounds. Suppose a win for you corresponds to H.
- Then, your net winnings are:

$$
\begin{array}{ll}
TTT & -\$7 \\
TTH & +\$1 \\
THT & +\$0 \\
THH & +\$2 \\
HTT & -\$2 \\
HTH & +\$2 \\
HHT & +\$1 \\
HHH & +\$3
\end{array}
$$

- Therefore, if M_3 denotes your winnings after 3 rounds of the game using the martingale betting strategy, then

$$\mathbb{E}[M_3] = 0.$$

Martingale transforms

- Is there a smarter way of varying our bets?
- We can formally capture betting strategies using the notion of **predictable sequences**.
- A sequence of random variables $A_1, A_2, \ldots$ is called predictable with respect to the sequence $X_1, X_2, \ldots$ if for all $n \geq 1$,

$$A_n = g_n(X_1, \ldots, X_{n-1}).$$

- So, if we think of $X_1, X_2, \ldots$ as being the outcomes of rounds of a gambling game, then A_n is a function of the information that the gambler has *before* placing the bet in the n^{th} round.

Martingale transforms

- Let $M_0, M_1, \ldots$ be a martingale with respect to $X_1, X_2, \ldots$, and let $A_1, A_2, \ldots$ be a predictable sequence with respect to $X_1, X_2, \ldots$.
- The **martingale transform** of $\{M_n\}$ by $\{A_n\}$ is defined by $\widetilde{M}_0 = M_0$ and for $n \geq 1$,

$$\widetilde{M}_n = M_0 + A_1(M_1 - M_0) + A_2(M_2 - M_1) + \cdots + A_n(M_n - M_{n-1}).$$

- Intuition: $(M_k - M_{k-1})$ is the gain from the k^{th} round of the gambling game. The gambler looks at all previous outcomes $X_1, \ldots, X_{k-1}$, and comes up with a multiplier A_k for the k^{th} round.

Martingale transforms are martingales

- Let $M_0, M_1, \ldots$ be a martingale with respect to $X_1, X_2, \ldots$, and let $A_1, A_2, \ldots$ be a predictable sequence with respect to $X_1, X_2, \ldots$.
- Let $\widetilde{M}_0, \widetilde{M}_1, \ldots$ be the martingale transform of $\{M_n\}$ by $\{A_n\}$.
- Then, $\widetilde{M}_0, \widetilde{M}_1, \ldots$ is also a martingale with respect to $X_1, X_2, \ldots$.
- Indeed,

$$\begin{aligned}
\mathbb{E}[\widetilde{M}_n - \widetilde{M}_{n-1} \mid X_1, \ldots, X_{n-1}] &= \mathbb{E}[A_n(M_n - M_{n-1}) \mid X_1, \ldots, X_{n-1}] \\
&= A_n \cdot \mathbb{E}[M_n - M_{n-1} \mid X_1, \ldots, X_{n-1}] \\
&= 0.
\end{aligned}$$