this week: martingales

STATS 217: Introduction to Stochastic Processes I

Lecture 26

Martingales

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- We say that the sequence of random variables $M_0, M_1, \ldots$ is a **martingale** with respect to $X_1, X_2, \ldots$ if

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  - $\mathbb{E}[M_n \mid X_1, X_2, \ldots, X_{n-1}] = M_{n-1}$. Explicitly, for any $x_1, \ldots, x_{n-1}$,

$$\mathbb{E}[M_n \mid X_1 = x_1, \ldots, X_{n-1} = x_{n-1}] = f_{n-1}(x_1, \ldots, x_{n-1}).$$

$\Longleftrightarrow \{ \mathbb{E}[M_n - M_{n-1} \mid X_1 \ldots X_{n-1}] = 0 \}$

$\Longleftrightarrow \mathbb{E}[M_n \mid X_1 \ldots X_{n-1}] = \mathbb{E}[M_{n-1} \mid X_1 \ldots X_{n-1}]$

Example

$\mathbb{E}[M_{n-1} \mid X_1 = x_1 \ldots X_{n-1} = x_{n-1}] = f_{n-1}(X_1 \ldots X_{n-1})$

$= f_{n-1}(x_1 \ldots x_{n-1})$

$= M_{n-1}$

e.g. simple symmetric R.W. in 1-D

then $X_i = \begin{cases} +1 & \text{w.p. } 1/2 \\ -1 & \text{w.p. } 1/2 \end{cases}$

- $X_1, X_2, \ldots$ are independent random variables with $\mathbb{E}[X_i] = 0$ for all $i \geq 1$.
- Let $M_0 = 0$ and for $n \geq 1$, $\mathbb{E}[|X_i|] < \infty$

$$M_n = X_1 + \cdots + X_n.$$

Example

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- Let $M_0 = 0$ and for $n \geq 1$, $\mathbb{E}[|X_i|] < \infty$

$$M_n = X_1 + \cdots + X_n.$$

- Then, $M_0, M_1, \ldots$ is a martingale with respect to $X_1, X_2, \ldots$.

(1) $\mathbb{E}(|M_n|) \leq \mathbb{E}[|X_1|] + \cdots + \mathbb{E}[|X_n|] < \infty.$

(2) $M_n = f(X_1, \ldots, X_n)$

(3) $\mathbb{E}[M_n \mid X_1 \ldots X_{n-1}] = M_{n-1}$

$\Leftrightarrow \mathbb{E}[\underbrace{M_n - M_{n-1}}_{X_n} \mid X_1 \ldots X_{n-1}] = \mathbb{E}[X_n \mid X_1 \ldots X_{n-1}]$
$= \mathbb{E}[X_n] = 0.$

Example

- $X_1, X_2, \ldots$ are independent random variables with $\mathbb{E}[X_i] = 0$ for all $i \geq 1$.
- Let $M_0 = 0$ and for $n \geq 1$,

$$M_n = X_1 + \cdots + X_n.$$

- Then, $M_0, M_1, \ldots$ is a martingale with respect to $X_1, X_2, \ldots$.
- This generalizes the one-dimensional simple, symmetric random walk.

Example

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- Then, $M_0 = 0$ and for $n \geq 1$,

$$M_n = (X_1 + \cdots + X_n)^2 - n\sigma^2 = S_n^2 - n\sigma^2$$

is a martingale with respect to $X_1, X_2, \ldots$.

* $M_n = f_n(X_1, \ldots, X_n)$

* $\mathbb{E}[M_n - M_{n-1} \mid X_1 \ldots X_{n-1}] = 0$.

Example

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- Then, $M_0 = 0$ and for $n \geq 1$,

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is a martingale with respect to $X_1, X_2, \ldots$.

- To verify the martingale property, note that

$$\mathbb{E}[M_n - M_{n-1} \mid X_1, \ldots, X_{n-1}] = \mathbb{E}[(X_n + S_{n-1})^2 - S_{n-1}^2 - \sigma^2 \mid X_1, \ldots, X_{n-1}]$$

$(X_n + S_{n-1})^2 - n\sigma^2$

$M_{n-1} = S_{n-1}^2 - (n-1)\sigma^2$

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$= \mathbb{E}[X_n^2 \mid X_1 \ldots X_{n-1}] - \sigma^2$

$\mathbb{E}[X_n^2] = \text{Var}(X_n) = \sigma^2$

$+ 2\,\mathbb{E}[X_n S_{n-1} \mid X_1 \ldots X_{n-1}]$

$= 2 S_{n-1}\, \mathbb{E}[X_n \mid X_1 \ldots X_{n-1}]$

Example

$= 2S_{n-1}\mathbb{E}[X_n] = 0.$

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&= 2S_{n-1}\mathbb{E}[X_n \mid X_1, \ldots, X_{n-1}] \\
&= 0
\end{aligned}$$

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- Then, $M_0 = 1$ and for $n \geq 1$,

$$M_n = M_0 \cdot X_1 \cdots X_n$$

is a martingale with respect to $X_1, \ldots, X_n$.

$$\mathbb{E}[M_n \mid X_1 \ldots X_{n-1}]$$
$$= \mathbb{E}[M_{n-1} X_n \mid X_1 \ldots X_{n-1}]$$
$$= M_{n-1} \mathbb{E}[X_n] = 1 \cdot M_{n-1} = M_{n-1}$$

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- Let $X_i = e^{\lambda Y_i}/\phi(\lambda)$. Then, $X_1, X_2, \ldots$ are independent random variables with $\mathbb{E}[X_i] = 1$.

$$\frac{\mathbb{E}[e^{\lambda Y_i}]}{\phi(\lambda)} = \frac{\phi(\lambda)}{\phi(\lambda)} = 1.$$

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- Let $X_i = e^{\lambda Y_i}/\phi(\lambda)$. Then, $X_1, X_2, \ldots$ are independent random variables with $\mathbb{E}[X_i] = 1$.
- Therefore, $M_0 = 1$ and for $n \geq 1$,

$$M_n = M_0 \cdot X_1 \cdots X_n = e^{\lambda(Y_1 + \cdots + Y_n)}/\phi(\lambda)^n$$

is a martingale with respect to $Y_1, Y_2, \ldots$.

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where $\xi_1, \xi_2, \ldots$ are i.i.d. copies of ξ.

- The sequence $M_0 = 1$ and for $n \geq 1$,

$$M_n = \frac{Z_n}{\mu^n}$$

is a martingale with respect to $M_1, M_2, \ldots$.

$$\mathbb{E}[M_n \mid Z_0 \ldots Z_{n-1}]$$
$$= \mathbb{E}[M_n \mid Z_{n-1}]$$
$$= \frac{1}{\mu^n} \mathbb{E}[Z_n \mid Z_{n-1}]$$
$$= \frac{1}{\mu^n} [\mu \cdot Z_{n-1}]$$
$$= \frac{Z_{n-1}}{\mu^{n-1}} = M_{n-1}.$$

Submartingales and supermartingales

- A **supermartingale** is defined similarly to a martingale, except now we weaken the martingale condition to

$$\mathbb{E}[M_n \mid X_1, \ldots, X_{n-1}] \leq M_{n-1}.$$

(=) for m.g.

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- Thinking of X_i as being the outcome of the i^{th} round of the gambling game, and M_n as being the wealth of the gambler after n rounds of the game, supermartingales are games that are unfavorable to the gambler.

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- On the other hand, **submartingales** are favorable to the gambler i.e., they satisfy

$$\mathbb{E}[M_n \mid X_1, \ldots, X_{n-1}] \geq M_{n-1}.$$

Martingale betting strategy

in the background: ind. fair coin tosses

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- You adopt the following strategy: if you win a round, then in the next round, you bet \$1; if you lose a round, then in the next round, you double your bet.

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- More generally, if you lose the first k rounds and win the $k+1^{st}$ round, your net winnings are

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- Moreover, in an infinite sequence of fair coin tosses, you will win with probability 1.

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- Then, your net winnings are:

TTT	$-\$7$
TTH	$+\$1$
THT	$+\$0$
THH	$+\$2$
HTT	$-\$2$
HTH	$+\$2$
HHT	$+\$1$
HHH	$+\$3$

TTT, TTH: -6

HTH, HHT, HHH: $+6$

Total: 0

Martingale betting strategy

- Let's take a look at this game for a fixed number of rounds, say 3 rounds. Suppose a win for you corresponds to H.
- Then, your net winnings are:

$$
\begin{array}{ll}
TTT & -\$7 \\
TTH & +\$1 \\
THT & +\$0 \\
THH & +\$2 \\
HTT & -\$2 \\
HTH & +\$2 \\
HHT & +\$1 \\
HHH & +\$3
\end{array}
$$

- Therefore, if M_3 denotes your winnings after 3 rounds of the game using the martingale betting strategy, then

$$\mathbb{E}[M_3] = 0.$$

Martingale transforms

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- A sequence of random variables $A_1, A_2, \ldots$ is called predictable with respect to the sequence $X_1, X_2, \ldots$ if for all $n \geq 1$,

$$A_n = g_n(X_1, \ldots, X_{n-1}).$$

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- So, if we think of $X_1, X_2, \ldots$ as being the outcomes of rounds of a gambling game, then A_n is a function of the information that the gambler has *before* placing the bet in the n^{th} round.

Martingale transforms

- Let $M_0, M_1, \ldots$ be a martingale with respect to $X_1, X_2, \ldots$, and let $A_1, A_2, \ldots$ be a predictable sequence with respect to $X_1, X_2, \ldots$.

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- The **martingale transform** of $\{M_n\}$ by $\{A_n\}$ is defined by $\widetilde{M}_0 = M_0$ and for $n \geq 1$,

$$\widetilde{M}_n = M_0 + A_1(M_1 - M_0) + A_2(M_2 - M_1) + \cdots + A_n(M_n - M_{n-1}).$$

↑ this records your wealth after n rounds of using your clever strategy.

$M_0 = 0$

(1) e.g. $M_n = X_1 + \ldots + X_n$

↓

$\tilde{M}_n = M_0 + A_1 X_1 + A_2 X_2 + \cdots + A_n X_n$, where $M_0 = 0$

$A_n = f_n(X_1 \ldots X_{n-1})$

Martingale transforms

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$$\widetilde{M}_n = M_0 + A_1(M_1 - M_0) + A_2(M_2 - M_1) + \cdots + A_n(M_n - M_{n-1}).$$

→ martingale differences

- Intuition: $(M_k - M_{k-1})$ is the gain from the k^{th} round of the gambling game. The gambler looks at all previous outcomes $X_1, \ldots, X_{k-1}$, and comes up with a multiplier A_k for the k^{th} round.

Martingale transforms are martingales

- Let $M_0, M_1, \ldots$ be a martingale with respect to $X_1, X_2, \ldots$, and let $A_1, A_2, \ldots$ be a predictable sequence with respect to $X_1, X_2, \ldots$.
- Let $\widetilde{M}_0, \widetilde{M}_1, \ldots$ be the martingale transform of $\{M_n\}$ by $\{A_n\}$.

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- Let $\widetilde{M}_0, \widetilde{M}_1, \ldots$ be the martingale transform of $\{M_n\}$ by $\{A_n\}$.
- Then, $\widetilde{M}_0, \widetilde{M}_1, \ldots$ is also a martingale with respect to $X_1, X_2, \ldots$.

$$\tilde{M}_n = f_n(X_1 \ldots X_n)$$

$$\mathbb{E}[\tilde{M}_n - \tilde{M}_{n-1} \mid X_1 \ldots X_{n-1}] = 0.$$

$$\tilde{M}_n - \tilde{M}_{n-1} = A_n \cdot (M_n - M_{n-1})$$

$$\mathbb{E}[A_n \cdot (M_n - M_{n-1}) \mid X_1 \ldots X_{n-1}] = A_n \, \mathbb{E}[M_n - M_{n-1} \mid X_1 \ldots X_{n-1}]$$
$$= A_n \cdot 0 = 0$$

Martingale transforms are martingales

- Let $M_0, M_1, \ldots$ be a martingale with respect to $X_1, X_2, \ldots$, and let $A_1, A_2, \ldots$ be a predictable sequence with respect to $X_1, X_2, \ldots$.
- Let $\widetilde{M}_0, \widetilde{M}_1, \ldots$ be the martingale transform of $\{M_n\}$ by $\{A_n\}$.
- Then, $\widetilde{M}_0, \widetilde{M}_1, \ldots$ is also a martingale with respect to $X_1, X_2, \ldots$.
- Indeed,

$$\begin{aligned}\mathbb{E}[\widetilde{M}_n - \widetilde{M}_{n-1} \mid X_1, \ldots, X_{n-1}] &= \mathbb{E}[A_n(M_n - M_{n-1}) \mid X_1, \ldots, X_{n-1}] \\ &= A_n \cdot \mathbb{E}[M_n - M_{n-1} \mid X_1, \ldots, X_{n-1}] \\ &= 0.\end{aligned}$$