STATS 217: Introduction to Stochastic Processes I

Lecture 27

Last time: martingale transforms

- Let $M_0, M_1, \ldots$ be a martingale with respect to $X_1, X_2, \ldots$, and let $A_1, A_2, \ldots$ be a predictable sequence with respect to $X_1, X_2, \ldots$.
- The **martingale transform** of $\{M_n\}$ by $\{A_n\}$ is defined by $\widetilde{M}_0 = M_0$ and for $n \geq 1$,

$$\widetilde{M}_n = M_0 + A_1(M_1 - M_0) + A_2(M_2 - M_1) + \cdots + A_n(M_n - M_{n-1}).$$

- Intuition: $(M_k - M_{k-1})$ is the gain from the k^{th} round of the gambling game. The gambler looks at all previous outcomes $X_1, \ldots, X_{k-1}$, and comes up with a multiplier A_k for the k^{th} round.

Last time: martingale transforms are martingales

- Let $M_0, M_1, \ldots$ be a martingale with respect to $X_1, X_2, \ldots$, and let $A_1, A_2, \ldots$ be a predictable sequence with respect to $X_1, X_2, \ldots$.
- Let $\widetilde{M}_0, \widetilde{M}_1, \ldots$ be the martingale transform of $\{M_n\}$ by $\{A_n\}$.
- Then, $\widetilde{M}_0, \widetilde{M}_1, \ldots$ is also a martingale with respect to $X_1, X_2, \ldots$.
- Indeed,

$$\begin{aligned}
\mathbb{E}[\widetilde{M}_n - \widetilde{M}_{n-1} \mid X_1, \ldots, X_{n-1}] &= \mathbb{E}[A_n(M_n - M_{n-1}) \mid X_1, \ldots, X_{n-1}] \\
&= A_n \cdot \mathbb{E}[M_n - M_{n-1} \mid X_1, \ldots, X_{n-1}] \\
&= 0.
\end{aligned}$$

Stopped martingales are martingales

- Recall that a stopping time with respect to $X_0, X_1, X_2, \ldots$ is a random variable τ taking values in $\{0, 1, 2, \ldots\} \cup \{\infty\}$ if for all $0 \leq n$, the event $\{\tau \leq n\}$ is determined by $X_0, \ldots, X_n$ i.e.,

$$\mathbb{1}_{\tau \leq n} = f_n(X_0, \ldots, X_n).$$

- Note that if τ is a stopping time, then

$$\mathbb{1}_{\tau \geq n} = 1 - \tau_{\leq n-1} = g_{n-1}(X_0, \ldots, X_{n-1}).$$

- Let $M_0, M_1, \ldots$ be a martingale with respect to $X_1, X_2, \ldots$ and let τ be a stopping time with respect to $X_0 = M_0, X_1, X_2, \ldots$. Then, the **stopped process** $M_{\min(0,\tau)}, M_{\min(1,\tau)}, \ldots$ is also a martingale with respect to $X_1, X_2, \ldots$.

Stopped martingales are martingales

- To see this, note that

$$M_{\min(n,\tau)} = M_n \mathbb{1}_{\tau \geq n} + M_\tau \mathbb{1}_{\tau \leq n-1}$$
$$= M_0 + \sum_{k=1}^{n} \mathbb{1}_{\tau \geq k} \cdot (M_k - M_{k-1}).$$

- Since $\mathbb{1}_{\tau \geq k} = g_{k-1}(X_0, \ldots, X_{k-1})$, it follows that

$$\widetilde{M}_n = M_{\min(n,\tau)}$$

is the martingale transform of $M_0, M_1, \ldots$ by the predictable sequence $A_k = \mathbb{1}_{\tau \geq k}$, and hence, is also a martingale.

Example: Gambler's ruin revisited

- Consider the simple symmetric random walk on the integers starting from 0 and with steps $X_1, X_2, \dots$.
- Let $M_0 = 0$ and $M_n = X_1 + \cdots + X_n$. Then, M_n is a martingale with respect to $X_1, X_2, \dots$.
- Let τ denote the first time that the walk visits A or -B.
- In the first lecture, we saw that $\mathbb{E}[\tau] < \infty$ and that

$$\mathbb{P}[M_\tau = A] = \frac{B}{A+B}.$$

- Here's another way to see this. Since $\widetilde{M}_n = M_{\min(n,\tau)}$ is a martingale, we must have

$$\mathbb{E}[\widetilde{M}_n] = \mathbb{E}[\mathbb{E}[\widetilde{M}_n \mid \widetilde{M}_{n-1}]] = \mathbb{E}[\widetilde{M}_{n-1}].$$

Example: Gambler's ruin revisited

- Therefore, by iteration,

$$\mathbb{E}[M_{\min(n,\tau)}] = 0$$

and since $\mathbb{P}[\tau < \infty] = 1$ and $|\widetilde{M}_n| \leq \max(A, B)$, we can take the limit as $n \to \infty$ to get that

$$\mathbb{E}[M_\tau] = 0$$

- On the other hand, we have

$$\begin{aligned} \mathbb{E}[M_\tau] &= A \cdot \mathbb{P}[M_\tau = A] - B \cdot \mathbb{P}[M_\tau = -B] \\ &= (A + B) \cdot \mathbb{P}[M_\tau = A] - B. \end{aligned}$$

- Combining these two equations, we get that

$$\mathbb{P}[M_\tau = A] = \frac{B}{A + B}.$$

- As an exercise, you can recover the result for the biased case by starting with the martingale $M_n = (q/p)^{X_1 + \cdots + X_n}$.

Example: Gambler's ruin revisited

- We also saw that $\mathbb{E}[\tau] = AB$.
- This can also be proved using a martingale argument. Recall from last time that $M_0 = 0$ and for $n \geq 1$,

$$M_n = (X_1 + \cdots + X_n)^2 - n$$

is a martingale.
- As before, we consider the stopped martingale and note that

$$\mathbb{E}[M_{\min(n,\tau)}] = 0.$$

- Using $\mathbb{E}[\tau] < \infty$, we can again take the limit as $n \to \infty$ to conclude that

$$\mathbb{E}[M_\tau] = 0.$$

Example: Gambler's ruin revisited

- On the other hand,

$$\begin{aligned}\mathbb{E}[M_\tau] = \mathbb{E}[M_\tau \mid (X_1 + \cdots + X_\tau) = A] \cdot \mathbb{P}[X_1 + \cdots + X_\tau = A] + \\ \mathbb{E}[M_\tau \mid (X_1 + \cdots + X_\tau) = B] \cdot \mathbb{P}[X_1 + \cdots + X_\tau = B] \\ = A^2 \cdot \frac{B}{A+B} + B^2 \cdot \frac{A}{A+B} - \mathbb{E}[\tau]\end{aligned}$$

- Setting the right hand side to 0 gives

$$\mathbb{E}[\tau] = AB.$$

Example: a card game

Consider the following card game.:

- There is a randomly shuffled deck of 52 cards, 26 of which are red, and 26 of which are black.
- The dealer deals one card at a time, face up.
- You are allowed to interject at most once to say that the next card is red.
- If the next card is indeed red, then you win $1. If the next card is black, you win nothing.
- What is the optimal expected payoff? What is a strategy achieving this payoff?

Example: a card game

- Formally, let the revealed cards be $X_1, X_2, \ldots, X_{52}$.
- Your goal is to come up with a stopping time τ with respect to $X_0 = 0, X_1, X_2, \ldots, X_{52}$ in order to maximize

$$\mathbb{E}[\mathbb{P}[X_{\tau+1} = \text{red} \mid X_1, \ldots, X_\tau]].$$

- If you set $\tau = 0$ (i.e., you always guess that the first card is red), then clearly,

$$\mathbb{E}[\mathbb{P}[X_{\tau+1} = \text{red} \mid X_1, \ldots, X_\tau]] = \mathbb{P}[X_1 = \text{red}] = 1/2.$$

- Can you do better? No!

Example: a card game

- Note that $\mathbb{P}[X_{\tau+1} = \text{red} \mid X_1, \ldots, X_\tau] = \mathbb{P}[X_{52} = \text{red} \mid X_1, \ldots, X_\tau]$.
- Therefore, our goal can be rephrased as trying to maximize

$$\mathbb{E}[M_\tau],$$

where $M_0 = 1/2$ and for $n \geq 1$,

$$M_n = \mathbb{P}[X_{52} = \text{red} \mid X_1, \ldots, X_n].$$

- Since

$$\mathbb{E}[\mathbb{P}[X_{52} = \text{red} \mid X_1, \ldots, X_n] \mid X_1, \ldots, X_{n-1}] = \mathbb{P}[X_{52} = \text{red} \mid X_1, \ldots, X_{n-1}],$$

it follows that M_n is a martingale. This is an example of a **Doob martingale**.

Example: a card game

- Therefore, $M_{\min(n,\tau)}$ is also a martingale.
- Since $\tau \leq 51$, it follows that

$$\begin{aligned}\mathbb{E}[M_\tau] &= \mathbb{E}[M_{\min(\tau,51)}] \\ &= \mathbb{E}[M_{\min(\tau,0)}] \\ &= \mathbb{E}[M_0] \\ &= \mathbb{P}[X_{52} = \text{red}] \\ &= 1/2.\end{aligned}$$