STATS 217: Introduction to Stochastic Processes I

Lecture 27

Last time: martingale transforms

- Let $M_0, M_1, \dots$ be a martingale with respect to $X_1, X_2, \dots$, and let $A_1, A_2, \dots$ be a predictable sequence with respect to $X_1, X_2, \dots$.
- The **martingale transform** of $\{M_n\}$ by $\{A_n\}$ is defined by $\widetilde{M}_0 = M_0$ and for $n \geq 1$,

$$\widetilde{M}_n = M_0 + A_1(M_1 - M_0) + A_2(M_2 - M_1) + \cdots + A_n(M_n - M_{n-1}).$$

- Intuition: $(M_k - M_{k-1})$ is the gain from the k^{th} round of the gambling game. The gambler looks at all previous outcomes $X_1, \dots, X_{k-1}$, and comes up with a multiplier A_k for the k^{th} round.

Last time: martingale transforms are martingales

- Let $M_0, M_1, \ldots$ be a martingale with respect to $X_1, X_2, \ldots$, and let $A_1, A_2, \ldots$ be a predictable sequence with respect to $X_1, X_2, \ldots$.
- Let $\widetilde{M}_0, \widetilde{M}_1, \ldots$ be the martingale transform of $\{M_n\}$ by $\{A_n\}$.
- Then, $\widetilde{M}_0, \widetilde{M}_1, \ldots$ is also a martingale with respect to $X_1, X_2, \ldots$.
- Indeed,

$$\begin{aligned}\mathbb{E}[\widetilde{M}_n - \widetilde{M}_{n-1} \mid X_1, \ldots, X_{n-1}] &= \mathbb{E}[A_n(M_n - M_{n-1}) \mid X_1, \ldots, X_{n-1}] \\ &= A_n \cdot \mathbb{E}[M_n - M_{n-1} \mid X_1, \ldots, X_{n-1}] \\ &= 0.\end{aligned}$$

$$\mathbb{E}[\widetilde{M}_n] = \mathbb{E}(\mathbb{E}[\widetilde{M}_n \mid \widetilde{M}_{n-1}])$$
$$= \mathbb{E}[\widetilde{M}_{n-1}]$$ iterate this $\mathbb{E}[\widetilde{M}_n] = \mathbb{E}[\widetilde{M}_0] = M_0.$

Stopped martingales are martingales

" optional stopping theorem"

- Recall that a stopping time with respect to $X_0, X_1, X_2, \dots$ is a random variable τ taking values in $\{0, 1, 2, \dots\} \cup \{\infty\}$ if for all $0 \leq n$, the event $\{\tau \leq n\}$ is determined by $X_0, \dots, X_n$ i.e.,

$$\mathbb{1}_{\tau \leq n} = f_n(X_0, \dots, X_n).$$

Stopped martingales are martingales

- Recall that a stopping time with respect to $X_0, X_1, X_2, \ldots$ is a random variable τ taking values in $\{0, 1, 2, \ldots\} \cup \{\infty\}$ if for all $0 \leq n$, the event $\{\tau \leq n\}$ is determined by $X_0, \ldots, X_n$ i.e.,

$$\mathbb{1}_{\tau \leq n} = f_n(X_0, \ldots, X_n).$$

- Note that if τ is a stopping time, then

$$\mathbb{1}_{\tau \geq n} = 1 - \mathbb{1}_{\tau \leq n-1} = g_{n-1}(X_0, \ldots, X_{n-1}).$$

Stopped martingales are martingales

- Recall that a stopping time with respect to $X_0, X_1, X_2, \ldots$ is a random variable τ taking values in $\{0, 1, 2, \ldots\} \cup \{\infty\}$ if for all $0 \leq n$, the event $\{\tau \leq n\}$ is determined by $X_0, \ldots, X_n$ i.e.,

$$\mathbb{1}_{\tau \leq n} = f_n(X_0, \ldots, X_n).$$

- Note that if τ is a stopping time, then

$$\mathbb{1}_{\tau \geq n} = 1 - \tau_{\leq n-1} = g_{n-1}(X_0, \ldots, X_{n-1}).$$

- Let $M_0, M_1, \ldots$ be a martingale with respect to $X_1, X_2, \ldots$ and let τ be a stopping time with respect to $X_0 = M_0, X_1, X_2, \ldots$.

Stopped martingales are martingales

- Recall that a stopping time with respect to $X_0, X_1, X_2, \ldots$ is a random variable τ taking values in $\{0, 1, 2, \ldots\} \cup \{\infty\}$ if for all $0 \leq n$, the event $\{\tau \leq n\}$ is determined by $X_0, \ldots, X_n$ i.e.,

$$\mathbb{1}_{\tau \leq n} = f_n(X_0, \ldots, X_n).$$

- Note that if τ is a stopping time, then

$$\mathbb{1}_{\tau \geq n} = 1 - \tau_{\leq n-1} = g_{n-1}(X_0, \ldots, X_{n-1}).$$

- Let $M_0, M_1, \ldots$ be a martingale with respect to $X_1, X_2, \ldots$ and let τ be a stopping time with respect to $X_0 = M_0, X_1, X_2, \ldots$. Then, the **stopped process** $M_{\min(0,\tau)}, M_{\min(1,\tau)}, \ldots$ is also a martingale with respect to $X_1, X_2, \ldots$.

$$\tilde{M}_n = M_{\min(n,\tau)} = \begin{cases} M_\tau : & \tau \geq n \\ M_n : & n < \tau \end{cases}$$

Stopped martingales are martingales

idea: stopped m.g. is the m.g. transform of $M_0, M_1, \ldots$ by a predictable sequence.

- To see this, note that

$$\tilde{M}_n = M_{\min(n,\tau)} = M_n \mathbb{1}_{\tau \geq n} + M_\tau \mathbb{1}_{\tau \leq n-1}$$

wts. that RHS is a martingale transform

$$= M_0 + A_1 (M_1 - M_0) + \cdots + A_n (M_n - M_{n-1})$$

where $A_1, A_2, \ldots$ is a predictable sequence.

$$A_k = g_{k-1}(M_0, X_1, \ldots, X_{k-1}).$$

claim: $A_k = \mathbb{1}_{[\tau \geq k]} \left(= 1 - \mathbb{1}_{[\tau \leq k-1]} \right)$

works.

Stopped martingales are martingales

- To see this, note that

$$
\begin{aligned}
M_{\min(n,\tau)} &= M_n \mathbb{1}_{\tau \geq n} + M_\tau \mathbb{1}_{\tau \leq n-1} \\
&= M_0 + \sum_{k=1}^{n} \mathbb{1}_{\tau \geq k} \cdot (M_k - M_{k-1}).
\end{aligned}
$$

$\tau = 3$

$$
\begin{aligned}
M_{\min(4,\tau)} &= M_0 + (M_1 - M_0) + (M_2 - M_1) \\
&\quad + (M_3 - M_2) = M_3 = M_\tau. \\
&\quad + \underbrace{\mathbb{1}_{\tau \geq 4}}_{0} (M_4 - M_3).
\end{aligned}
$$

Stopped martingales are martingales

- To see this, note that

$$\begin{aligned} M_{\min(n,\tau)} &= M_n \mathbb{1}_{\tau \geq n} + M_\tau \mathbb{1}_{\tau \leq n-1} \\ &= M_0 + \sum_{k=1}^{n} \mathbb{1}_{\tau \geq k} \cdot (M_k - M_{k-1}). \end{aligned}$$

- Since $\mathbb{1}_{\tau \geq k} = g_{k-1}(X_0, \ldots, X_{k-1})$, it follows that

$$\widetilde{M}_n = M_{\min(n,\tau)}$$

  is the martingale transform of $M_0, M_1, \ldots$ by the predictable sequence $A_k = \mathbb{1}_{\tau \geq k}$,

Stopped martingales are martingales

- To see this, note that

$$\begin{aligned} M_{\min(n,\tau)} &= M_n \mathbb{1}_{\tau \geq n} + M_\tau \mathbb{1}_{\tau \leq n-1} \\ &= M_0 + \sum_{k=1}^{n} \mathbb{1}_{\tau \geq k} \cdot (M_k - M_{k-1}). \end{aligned}$$

- Since $\mathbb{1}_{\tau \geq k} = g_{k-1}(X_0, \ldots, X_{k-1})$, it follows that

$$\widetilde{M}_n = M_{\min(n,\tau)}$$

is the martingale transform of $M_0, M_1, \ldots$ by the predictable sequence $A_k = \mathbb{1}_{\tau \geq k}$, and hence, is also a martingale.

Example: Gambler's ruin revisited

- Consider the simple symmetric random walk on the integers starting from 0 and with steps $X_1, X_2, \ldots$. each X_i is independently $\begin{cases} +1 & \text{w.p. } \frac{1}{2} \\ -1 & \text{w.p. } \frac{1}{2} \end{cases}$
- Let $M_0 = 0$ and $M_n = X_1 + \cdots + X_n$.

  (Recall: M_n is a m.g. wrt $X_1, X_2, \ldots$)

Example: Gambler's ruin revisited

- Consider the simple symmetric random walk on the integers starting from 0 and with steps $X_1, X_2, \ldots$.
- Let $M_0 = 0$ and $M_n = X_1 + \cdots + X_n$. Then, M_n is a martingale with respect to $X_1, X_2, \ldots$.

Example: Gambler's ruin revisited

- Consider the simple symmetric random walk on the integers starting from 0 and with steps $X_1, X_2, \ldots$.
- Let $M_0 = 0$ and $M_n = X_1 + \cdots + X_n$. Then, M_n is a martingale with respect to $X_1, X_2, \ldots$.
- Let τ denote the first time that the walk visits A or -B.

Example: Gambler's ruin revisited

- Consider the simple symmetric random walk on the integers starting from 0 and with steps $X_1, X_2, \ldots$.
- Let $M_0 = 0$ and $M_n = X_1 + \cdots + X_n$. Then, M_n is a martingale with respect to $X_1, X_2, \ldots$.
- Let τ denote the first time that the walk visits A or $-B$.
- In the first lecture, we saw that $\mathbb{E}[\tau] < \infty$ and that

$$\mathbb{P}[M_\tau = A] = \frac{B}{A+B}.$$

{ We proved this using first step analysis. }.

Example: Gambler's ruin revisited

- Consider the simple symmetric random walk on the integers starting from 0 and with steps $X_1, X_2, \ldots$.
- Let $M_0 = 0$ and $M_n = X_1 + \cdots + X_n$. Then, M_n is a martingale with respect to $X_1, X_2, \ldots$.
- Let τ denote the first time that the walk visits A or -B.
- In the first lecture, we saw that $\mathbb{E}[\tau] < \infty$ and that

$$\mathbb{P}[M_\tau = A] = \frac{B}{A + B}.$$

- Here's another way to see this.

Example: Gambler's ruin revisited

- Consider the simple symmetric random walk on the integers starting from 0 and with steps $X_1, X_2, \ldots$.
- Let $M_0 = 0$ and $M_n = X_1 + \cdots + X_n$. Then, M_n is a martingale with respect to $X_1, X_2, \ldots$.
- Let τ denote the first time that the walk visits A or -B.
- In the first lecture, we saw that $\mathbb{E}[\tau] < \infty$ and that

$$\mathbb{P}[M_\tau = A] = \frac{B}{A+B}.$$

- Here's another way to see this. Since $\widetilde{M}_n = M_{\min(n,\tau)}$ is a martingale, we must have

$$\mathbb{E}[\widetilde{M}_n] = \mathbb{E}[\mathbb{E}[\widetilde{M}_n \mid \widetilde{M}_{n-1}]] = \mathbb{E}[\widetilde{M}_{n-1}].$$

Example: Gambler's ruin revisited

- Therefore, by iteration,

$$\mathbb{E}[M_{\min(n,\tau)}] = 0$$

what we would like to show is that

$$\mathbb{E}[M_\tau] = 0.$$

since $\mathbb{P}[\tau < \infty] = 1$

$$\lim_{n\to\infty} M_{\min(n,\tau)} = M_\tau$$

$$\mathbb{E}[M_\tau] = \mathbb{E}[\lim_{n\to\infty} M_{\min(n,\tau)}]$$

$|M_{\min(n,\tau)}| \leq \max\{A, B\}$ $\longrightarrow$ $\overset{\bigcirc}{=}$ $\lim_{n\to\infty} \mathbb{E}[M_{\min(n,\tau)}] = 0.$

"dominated conv. thm"

Example: Gambler's ruin revisited

- Therefore, by iteration,

$$\mathbb{E}[M_{\min(n,\tau)}] = 0$$

and since $\mathbb{P}[\tau < \infty] = 1$ and $|\widetilde{M}_n| \leq \max(A, B)$, we can take the limit as $n \to \infty$

Example: Gambler's ruin revisited

- Therefore, by iteration,

$$\mathbb{E}[M_{\min(n,\tau)}] = 0$$

and since $\mathbb{P}[\tau < \infty] = 1$ and $|\widetilde{M}_n| \leq \max(A, B)$, we can take the limit as $n \to \infty$ to get that

$$\mathbb{E}[M_\tau] = 0$$

- how does this give you $\mathbb{P}[M_\tau = A]$?

$$\mathbb{E}[M_\tau] = \mathbb{E}[M_\tau \mid M_\tau = A]\,\mathbb{P}[M_\tau = A] + \mathbb{E}[M_\tau \mid M_\tau = -B]\,\mathbb{P}[M_\tau = -B]$$

$$= A \cdot \mathbb{P}[M_\tau = A] - B\,\mathbb{P}[M_\tau = -B]$$

Example: Gambler's ruin revisited

- Therefore, by iteration,

$$\mathbb{E}[M_{\min(n,\tau)}] = 0$$

and since $\mathbb{P}[\tau < \infty] = 1$ and $|\widetilde{M}_n| \leq \max(A, B)$, we can take the limit as $n \to \infty$ to get that

$$\mathbb{E}[M_\tau] = 0$$

- On the other hand, we have

$$\mathbb{E}[M_\tau] = A \cdot \mathbb{P}[M_\tau = A] - B \cdot \mathbb{P}[M_\tau = -B]$$

Example: Gambler's ruin revisited

- Therefore, by iteration,

$$\mathbb{E}[M_{\min(n,\tau)}] = 0$$

and since $\mathbb{P}[\tau < \infty] = 1$ and $|\widetilde{M}_n| \leq \max(A, B)$, we can take the limit as $n \to \infty$ to get that

$$\mathbb{E}[M_\tau] = 0$$

- On the other hand, we have

$$\begin{aligned} 0 = \quad \mathbb{E}[M_\tau] &= A \cdot \mathbb{P}[M_\tau = A] - B \cdot \mathbb{P}[M_\tau = -B] \\ &= (A + B) \cdot \mathbb{P}[M_\tau = A] - B. \end{aligned}$$

$$\Rightarrow \quad \mathbb{P}[M_\tau = A] = \frac{B}{A+B}.$$

Example: Gambler's ruin revisited

- Therefore, by iteration,
$$\mathbb{E}[M_{\min(n,\tau)}] = 0$$
and since $\mathbb{P}[\tau < \infty] = 1$ and $|\widetilde{M}_n| \leq \max(A, B)$, we can take the limit as $n \to \infty$ to get that
$$\mathbb{E}[M_\tau] = 0$$

- On the other hand, we have
$$\begin{aligned} \mathbb{E}[M_\tau] &= A \cdot \mathbb{P}[M_\tau = A] - B \cdot \mathbb{P}[M_\tau = -B] \\ &= (A + B) \cdot \mathbb{P}[M_\tau = A] - B. \end{aligned}$$

- Combining these two equations, we get that
$$\mathbb{P}[M_\tau = A] = \frac{B}{A + B}.$$

Example: Gambler's ruin revisited

- Therefore, by iteration,

$$\mathbb{E}[M_{\min(n,\tau)}] = 0$$

and since $\mathbb{P}[\tau < \infty] = 1$ and $|\widetilde{M}_n| \le \max(A, B)$, we can take the limit as $n \to \infty$ to get that

$$\mathbb{E}[M_\tau] = 0$$

- On the other hand, we have

$$\begin{aligned}\mathbb{E}[M_\tau] &= A \cdot \mathbb{P}[M_\tau = A] - B \cdot \mathbb{P}[M_\tau = -B] \\ &= (A + B) \cdot \mathbb{P}[M_\tau = A] - B.\end{aligned}$$

- Combining these two equations, we get that

$$\mathbb{P}[M_\tau = A] = \frac{B}{A + B}.$$

- As an exercise, you can recover the result for the biased case by starting with the martingale $M_n = (q/p)^{X_1 + \cdots + X_n}$.

Example: Gambler's ruin revisited

- We also saw that $\mathbb{E}[\tau] = AB$.

Example: Gambler's ruin revisited

- We also saw that $\mathbb{E}[\tau] = AB$.
- This can also be proved using a martingale argument.

Example: Gambler's ruin revisited

- We also saw that $\mathbb{E}[\tau] = AB$.
- This can also be proved using a martingale argument. Recall from last time that $M_0 = 0$ and for $n \geq 1$,

$$M_n = (X_1 + \cdots + X_n)^2 - n$$

is a martingale.

last time:

X_i have $\mathbb{E}[X_i] = 0$

$\mathrm{Var}(X_i) = \sigma^2$

$(X_1 + \cdots + X_n)^2 - n\sigma^2$

Example: Gambler's ruin revisited

- We also saw that $\mathbb{E}[\tau] = AB$.
- This can also be proved using a martingale argument. Recall from last time that $M_0 = 0$ and for $n \geq 1$,

$$M_n = (X_1 + \cdots + X_n)^2 - n$$

  is a martingale.
- As before, we consider the stopped martingale and note that

$$\mathbb{E}[M_{\min(n,\tau)}] = 0.$$

Example: Gambler's ruin revisited

- We also saw that $\mathbb{E}[\tau] = AB$.
- This can also be proved using a martingale argument. Recall from last time that $M_0 = 0$ and for $n \geq 1$,

$$M_n = (X_1 + \cdots + X_n)^2 - n$$

is a martingale.

- As before, we consider the stopped martingale and note that

$$\mathbb{E}[M_{\min(n,\tau)}] = 0.$$

- Using $\mathbb{E}[\tau] < \infty$, we can again take the limit as $n \to \infty$ to conclude that

$$\mathbb{E}[M_\tau] = 0.$$

Example: Gambler's ruin revisited

at $M_\tau \to X_1 + \cdots + X_\tau = A$

$\quad\quad \to X_1 + \cdots + X_\tau = -B$

- On the other hand,

$$\mathbb{E}[M_\tau] = \mathbb{E}[M_\tau \mid (X_1 + \cdots + X_\tau) = A] \cdot \mathbb{P}[X_1 + \cdots + X_\tau = A] + \mathbb{E}[M_\tau \mid (X_1 + \cdots + X_\tau) = -B] \cdot \mathbb{P}[X_1 + \cdots + X_\tau = -B]$$

$$= \mathbb{E}[M_\tau \mid X_1 + \cdots + X_\tau = A] \cdot \frac{B}{A+B}$$

$$+ \mathbb{E}[M_\tau \mid X_1 + \cdots + X_\tau = -B] \cdot \frac{A}{A+B}.$$

$$M_\tau = (X_1 + \cdots + X_\tau)^2 - \tau$$

Example: Gambler's ruin revisited

- On the other hand,

$$
\begin{aligned}
\mathbb{E}[M_\tau] = \mathbb{E}[M_\tau \mid (X_1 + \cdots + X_\tau) = A] \cdot \mathbb{P}[X_1 + \cdots + X_\tau = A] + \\
\mathbb{E}[M_\tau \mid (X_1 + \cdots + X_\tau) = B] \cdot \mathbb{P}[X_1 + \cdots + X_\tau = B] \\
= A^2 \cdot \frac{B}{A+B} + B^2 \cdot \frac{A}{A+B} - \mathbb{E}[\tau]
\end{aligned}
$$

$$= 0 \quad (\Leftarrow)$$

$$\mathbb{E}[\tau] = AB \frac{(A+B)}{(A+B)}$$

last time: $M_n = \dfrac{e^{\lambda(X_1 + \cdots + X_n)}}{(\mathbb{E}[e^{\lambda X_i}])^n}$

Example: Gambler's ruin revisited

- On the other hand,

$$\begin{aligned}\mathbb{E}[M_\tau] = \mathbb{E}[M_\tau \mid (X_1 + \cdots + X_\tau) = A] \cdot \mathbb{P}[X_1 + \cdots + X_\tau = A] + \\ \mathbb{E}[M_\tau \mid (X_1 + \cdots + X_\tau) = B] \cdot \mathbb{P}[X_1 + \cdots + X_\tau = B] \\ = A^2 \cdot \frac{B}{A+B} + B^2 \cdot \frac{A}{A+B} - \mathbb{E}[\tau]\end{aligned}$$

- Setting the right hand side to 0 gives

$$\mathbb{E}[\tau] = AB.$$

Example: a card game

Consider the following card game.:

- There is a randomly shuffled deck of 52 cards, 26 of which are red, and 26 of which are black.

Example: a card game

Consider the following card game.:

- There is a randomly shuffled deck of 52 cards, 26 of which are red, and 26 of which are black.
- The dealer deals one card at a time, face up.

Example: a card game

Consider the following card game.:

- There is a randomly shuffled deck of 52 cards, 26 of which are red, and 26 of which are black.
- The dealer deals one card at a time, face up.
- You are allowed to interject at most once to say that the next card is red.

Example: a card game

Consider the following card game.:

- There is a randomly shuffled deck of 52 cards, 26 of which are red, and 26 of which are black.
- The dealer deals one card at a time, face up.
- You are allowed to interject at most once to say that the next card is red.
- If the next card is indeed red, then you win \$1. If the next card is black, you win nothing.

you are betting when either

0 cards shown

1 card shown

⋮

51 cards shown.

Example: a card game

Consider the following card game.:

- There is a randomly shuffled deck of 52 cards, 26 of which are red, and 26 of which are black.
- The dealer deals one card at a time, face up.
- You are allowed to interject at most once to say that the next card is red.
- If the next card is indeed red, then you win $1. If the next card is black, you win nothing.
- What is the optimal expected payoff? What is a strategy achieving this payoff?

Example: a card game

- Formally, let the revealed cards be $X_1, X_2, \ldots, X_{52}$.

Example: a card game

- Formally, let the revealed cards be $X_1, X_2, \ldots, X_{52}$.
- Your goal is to come up with a stopping time τ with respect to $X_0 = 0, X_1, X_2, \ldots, X_{52}$ in order to maximize

$$\mathbb{E}[\mathbb{P}[X_{\tau+1} = \text{red} \mid X_1, \ldots, X_\tau]].$$

Example: a card game

- Formally, let the revealed cards be $X_1, X_2, \ldots, X_{52}$.
- Your goal is to come up with a stopping time τ with respect to $X_0 = 0, X_1, X_2, \ldots, X_{52}$ in order to maximize

$$\mathbb{E}[\mathbb{P}[X_{\tau+1} = \text{red} \mid X_1, \ldots, X_\tau]].$$

- If you set $\tau = 0$ (i.e., you always guess that the first card is red), then clearly,

$$\mathbb{E}[\mathbb{P}[X_{\tau+1} = \text{red} \mid X_1, \ldots, X_\tau]] = \mathbb{P}[X_1 = \text{red}] = 1/2.$$

Example: a card game

- Formally, let the revealed cards be $X_1, X_2, \ldots, X_{52}$.
- Your goal is to come up with a stopping time τ with respect to $X_0 = 0, X_1, X_2, \ldots, X_{52}$ in order to maximize

$$\mathbb{E}[\mathbb{P}[X_{\tau+1} = \text{red} \mid X_1, \ldots, X_\tau]].$$

- If you set $\tau = 0$ (i.e., you always guess that the first card is red), then clearly,

$$\mathbb{E}[\mathbb{P}[X_{\tau+1} = \text{red} \mid X_1, \ldots, X_\tau]] = \mathbb{P}[X_1 = \text{red}] = 1/2.$$

- Can you do better?

Example: a card game

- Formally, let the revealed cards be $X_1, X_2, \ldots, X_{52}$.
- Your goal is to come up with a stopping time τ with respect to $X_0 = 0, X_1, X_2, \ldots, X_{52}$ in order to maximize

$$\mathbb{E}[\mathbb{P}[X_{\tau+1} = \text{red} \mid X_1, \ldots, X_\tau]].$$

- If you set $\tau = 0$ (i.e., you always guess that the first card is red), then clearly,

$$\mathbb{E}[\mathbb{P}[X_{\tau+1} = \text{red} \mid X_1, \ldots, X_\tau]] = \mathbb{P}[X_1 = \text{red}] = 1/2.$$

- Can you do better? No!

Example: a card game

- Note that $\mathbb{P}[X_{\tau+1} = \text{red} \mid X_1, \ldots, X_\tau] = \mathbb{P}[X_{52} = \text{red} \mid X_1, \ldots, X_\tau]$.

Example: a card game

- Note that $\mathbb{P}[X_{\tau+1} = \text{red} \mid X_1, \ldots, X_\tau] = \mathbb{P}[X_{52} = \text{red} \mid X_1, \ldots, X_\tau]$.
- Therefore, our goal can be rephrased as trying to maximize

$$\mathbb{E}[M_\tau],$$

where $M_0 = 1/2$ and for $n \geq 1$,

$$M_n = \mathbb{P}[X_{52} = \text{red} \mid X_1, \ldots, X_n].$$

Example: a card game

- Note that $\mathbb{P}[X_{\tau+1} = \text{red} \mid X_1, \ldots, X_\tau] = \mathbb{P}[X_{52} = \text{red} \mid X_1, \ldots, X_\tau]$.
- Therefore, our goal can be rephrased as trying to maximize

$$\mathbb{E}[M_\tau],$$

where $M_0 = 1/2$ and for $n \geq 1$,

$$M_n = \mathbb{P}[X_{52} = \text{red} \mid X_1, \ldots, X_n].$$

- Since

$$\mathbb{E}[\mathbb{P}[X_{52} = \text{red} \mid X_1, \ldots, X_n] \mid X_1, \ldots, X_{n-1}] = \mathbb{P}[X_{52} = \text{red} \mid X_1, \ldots, X_{n-1}],$$

$x_1 \ldots x_{n-1}, x_n$

$M_n.$

$= M_{n-1}$

Example: a card game

- Note that $\mathbb{P}[X_{\tau+1} = \text{red} \mid X_1, \ldots, X_\tau] = \mathbb{P}[X_{52} = \text{red} \mid X_1, \ldots, X_\tau]$.
- Therefore, our goal can be rephrased as trying to maximize

$$\mathbb{E}[M_\tau],$$

where $M_0 = 1/2$ and for $n \geq 1$,

$$M_n = \mathbb{P}[X_{52} = \text{red} \mid X_1, \ldots, X_n].$$

- Since

$$\mathbb{E}[\mathbb{P}[X_{52} = \text{red} \mid X_1, \ldots, X_n] \mid X_1, \ldots, X_{n-1}] = \mathbb{P}[X_{52} = \text{red} \mid X_1, \ldots, X_{n-1}],$$

it follows that M_n is a martingale.

Example: a card game

- Note that $\mathbb{P}[X_{\tau+1} = \text{red} \mid X_1, \ldots, X_\tau] = \mathbb{P}[X_{52} = \text{red} \mid X_1, \ldots, X_\tau]$.
- Therefore, our goal can be rephrased as trying to maximize

$$\mathbb{E}[M_\tau],$$

where $M_0 = 1/2$ and for $n \geq 1$,

$$M_n = \mathbb{P}[X_{52} = \text{red} \mid X_1, \ldots, X_n].$$

- Since

$$\mathbb{E}[M_n \mid M_{n-1}] = M_{n-1}$$

$$\mathbb{E}[\mathbb{P}[X_{52} = \text{red} \mid X_1, \ldots, X_n] \mid X_1, \ldots, X_{n-1}] = \mathbb{P}[X_{52} = \text{red} \mid X_1, \ldots, X_{n-1}],$$

it follows that M_n is a martingale. This is an example of a **Doob martingale**.

Example: a card game

- Therefore, $M_{\min(n,\tau)}$ is also a martingale.

Example: a card game

- Therefore, $M_{\min(n,\tau)}$ is also a martingale.
- Since $\tau \leq 51$, it follows that

$$\mathbb{E}[M_\tau] = \mathbb{E}[M_{\min(\tau,51)}]$$

Example: a card game

- Therefore, $M_{\min(n,\tau)}$ is also a martingale.
- Since $\tau \leq 51$, it follows that

$$\begin{aligned}\mathbb{E}[M_\tau] &= \mathbb{E}[M_{\min(\tau,51)}] \\ &= \mathbb{E}[M_{\min(\tau,0)}]\end{aligned}$$

Example: a card game

- Therefore, $M_{\min(n,\tau)}$ is also a martingale.
- Since $\tau \leq 51$, it follows that

$$\begin{aligned} \mathbb{E}[M_\tau] &= \mathbb{E}[M_{\min(\tau,51)}] \\ &= \mathbb{E}[M_{\min(\tau,0)}] \\ &= \mathbb{E}[M_0] \end{aligned}$$

Example: a card game

- Therefore, $M_{\min(n,\tau)}$ is also a martingale.
- Since $\tau \leq 51$, it follows that

$$\begin{aligned} \mathbb{E}[M_\tau] &= \mathbb{E}[M_{\min(\tau,51)}] \\ &= \mathbb{E}[M_{\min(\tau,0)}] \\ &= \mathbb{E}[M_0] \\ &= \mathbb{P}[X_{52} = \text{red}] \end{aligned}$$

Example: a card game

- Therefore, $M_{\min(n,\tau)}$ is also a martingale.
- Since $\tau \leq 51$, it follows that

$$\begin{aligned}
\mathbb{E}[M_\tau] &= \mathbb{E}[M_{\min(\tau,51)}] \\
&= \mathbb{E}[M_{\min(\tau,0)}] \\
&= \mathbb{E}[M_0] \\
&= \mathbb{P}[X_{52} = \text{red}] \\
&= 1/2.
\end{aligned}$$

$M_n = \mathbb{P}(X_{52} = \text{red} \mid X_1 \dots X_n)$

$M_0 = \mathbb{P}(X_{52} = \text{Red} \mid X_0)$