STATS 217: Introduction to Stochastic Processes I

Lecture 28

Last time: stopped martingales are martingales

- Recall that a stopping time with respect to $X_0, X_1, X_2, \ldots$ is a random variable τ taking values in $\{0, 1, 2, \ldots\} \cup \{\infty\}$ if for all $0 \leq n$, the event $\{\tau \leq n\}$ is determined by $X_0, \ldots, X_n$ i.e.,

$$\mathbb{1}_{\tau \leq n} = f_n(X_0, \ldots, X_n).$$

- Let $M_0, M_1, \ldots$ be a martingale with respect to $X_1, X_2, \ldots$ and let τ be a stopping time with respect to $X_0 = M_0, X_1, X_2, \ldots$.

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- Let $M_0, M_1, \ldots$ be a martingale with respect to $X_1, X_2, \ldots$ and let τ be a stopping time with respect to $X_0 = M_0, X_1, X_2, \ldots$. Then, the **stopped process** $M_{\min(0,\tau)}, M_{\min(1,\tau)}, \ldots$ is also a martingale with respect to $X_1, X_2, \ldots$.
- This is called the **optional stopping theorem**.

(name due to Doob)

Example: waiting time for patterns

- Suppose a fair coin is tossed repeatedly. Let τ be the first time to get the pattern HTH. What is $\mathbb{E}[\tau]$?

$\tau - 2$ $\tau - 1$ τ

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  idea: gambler wants HTH
- At every time step, a *new* gambler joins the table and buys a chip for \$1.
- Initially, the gambler bets \$1 on the coin landing heads. If the gambler loses, she leaves with nothing.

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- At every time step, a *new* gambler joins the table and buys a chip for $1.
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- On Homework 5, you showed that $\mathbb{E}[\tau] = 10$ using first step analysis. Now, we'll see a quicker way to do this using the optional stopping theorem.
- For this, imagine a table at a casino where a fair coin is tossed at every time step.
- At every time step, a *new* gambler joins the table and buys a chip for $1.
- Initially, the gambler bets $1 on the coin landing heads. If the gambler loses, she leaves with nothing. If she wins, she now has $2 which she bets on the coin landing tails. If she loses, she leaves with nothing. If she wins, she now has $4, which she bets on the coin landing heads. If she loses, she leaves with nothing. If she wins, she leaves with $8.

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- Let $X_1, X_2, \ldots$ denote the outcomes of the coin tosses.
- Let $M_0 = 0$ and M_n denote the money the casino has after X_n has been revealed. Note that $M_n = f(X_1, \ldots, X_n)$.

- moreover:

$$\mathbb{E}[M_n \mid X_1 \ldots X_{n-1}] = M_{n-1}$$

note that stake on $(X_n = H) = g_H(X_1 \ldots X_{n-1})$

— on $(X_n = T) = g_T(X_1 \ldots X_{n-1})$

$$M_n = M_{n-1} + \mathbb{1}_{X_n = H}(g_T - g_H) + \mathbb{1}_{X_n = T}(g_H - g_T)$$

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- Moreover, since X_n is a fair coin, $\mathbb{E}[M_n \mid X_1, \ldots, X_{n-1}] = M_{n-1}$ so that M_n is a martingale.

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- Let τ denote the first time that the pattern HTH is observed at the table. Then, τ is a stopping time with respect to $X_1, X_2, \ldots$ so that by the optional stopping theorem, $\widetilde{M}_n = M_{\min(n,\tau)}$ is a martingale as well.

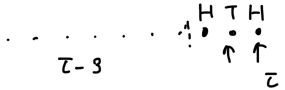
- Therefore,

$$\mathbb{E}[M_{\min(n,\tau)}] = \mathbb{E}[M_0] = 0.$$

- Using $\mathbb{E}[\tau] < \infty$, we are able to take (and switch) limits to deduce that

$$\mathbb{E}[M_\tau] = 0.$$

- We want to compute $\mathbb{E}[M_\tau]$ in a diff. way (hopefully involving $\mathbb{E}[\tau]$).



$$M_\tau = (\tau - 3 + 1) - 1 - 7 = \tau - 10$$

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- Therefore, $0 = \mathbb{E}[M_\tau] = (\mathbb{E}[\tau] - 2) - 1 - 7$,

$0 = \mathbb{E}[M_\tau] = \mathbb{E}[\tau] - 10 \Rightarrow \mathbb{E}[\tau] = 10.$

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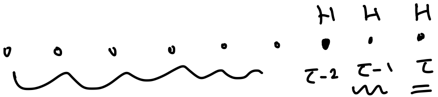
- Therefore, $0 = \mathbb{E}[M_\tau] = (\mathbb{E}[\tau] - 2) - 1 - 7$, so that $\mathbb{E}[\tau] = 10$.

Example: waiting time for patterns

- Similarly, if τ denotes the waiting time for HHH, then the casino loses \$1 to the gambler who enters at time τ, \$3 to the gambler who enters at time $\tau - 1$, and \$7 to the gambler who enters at time $\tau - 2$.



$$M_\tau = \tau - 3 - (7 + 3 + 1)$$

$$= \tau - 14$$

$$\mathbb{E}[M_\tau] = 0 \Rightarrow \mathbb{E}[\tau] = 14.$$

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- Moreover, the casino makes \$1 from the first $\tau - 3$ gamblers.
- Therefore, the same argument gives

$$(\mathbb{E}[\tau] - 3) - 1 - 3 - 7 = 0$$

so that $\mathbb{E}[\tau] = 14$.

Martingale convergence theorem

other forms of the conv. thm as well.

The martingale convergence theorem asserts that if $M_n \geq 0$ is a martingale, then there exists a random variable M_∞ such that

$$\lim_{n\to\infty} M_n = M_\infty \quad \text{(in an almost sure sense)}$$

$\mathbb{P}[\omega \in \Omega : \lim_{n\to\infty} M_n(\omega) = M_\infty(\omega)] = 1.$

and

$$\mathbb{E}[M_\infty] \leq \mathbb{E}[M_0].$$

why is this not $=$? (Fatou's lemma).

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- Also, even if $M_n \geq 0$ is a martingale, it is not necessarily true that

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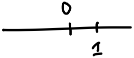

- For example, consider the simple, symmetric random walk on the integers starting at 1, and let τ be the first time that the walk visits 0. Then, the stopped martingale $\widetilde{M}_n = M_{\min(n,\tau)}$ satisfies $\widetilde{M}_n \geq 0$ and $\lim_{n\to\infty} \widetilde{M}_n = 0$.

Example: branching processes

- Let $(Z_n)_{n\geq 0}$ be a branching process with $Z_0 = 1$ and offspring distribution ξ.
- Recall this means that

$$Z_n = \sum_{i=1}^{Z_{n-1}} \xi_i,$$

where ξ_i are i.i.d. copies of ξ.

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- Let $\mu = \mathbb{E}[\xi] > 0$. We saw that

$$M_n = Z_n/\mu^n$$

is a martingale.

Example: branching processes

- Since M_n is a non-negative martingale, there exists a random variable M_∞ such that

$$\lim_{n\to\infty} M_n = M_\infty$$

and $\mathbb{E}[M_\infty] \leq \mathbb{E}[M_0] = 1$.

$\circ$ $\left\{ \begin{array}{l} \mu < 1 \\ \mu = 1 \end{array} \right\}$ and if $\mathbb{P}[\xi = 1] < 1 \Rightarrow \mathbb{P}[\text{extinction}] = 1$.

$\mu > 1$.

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- When $\mu = 1$, then $Z_n = M_n$ is a martingale. In this case, we saw that the probability of extinction is 1 (provided that $\mathbb{P}[\xi = 1] < 1$). Here's a martingale proof of this fact.

- $M_n = \frac{Z_n}{\mu^n}$ is a m.g.

but $\mu = 1$, $M_n = Z_n$ so Z_n is also a m.g.

so $\exists$ M_∞ s.t. $Z_n \longrightarrow M_\infty$ a.s.

& since Z_n is integer valued, M_∞ must also be int. valued.

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- Since Z_n is integer-valued and $Z_n \to M_\infty$, it must be the case that M_∞ is also integer-valued.
- We claim $\mathbb{P}[M_\infty > 0] = 0$. Otherwise, there would exist some $k \geq 1$ such that $\mathbb{P}[M_\infty = k] > 0$ and hence $\mathbb{P}[\exists N : Z_n = k \quad \forall n \geq N] > 0$.

• k
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⋮

$\mathbb{P}[\xi = 1] < 1$

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- But the last event has probability 0 if $\mathbb{P}[\xi = 1] < 1$.

Martingale convergence theorem

- Here's some intuition for the martingale convergence theorem. While $\lim_{n\to\infty} M_n$ need not exist, we can always talk about

$$Y := \liminf_{n\to\infty} M_n, \quad Z := \limsup_{n\to\infty} M_n.$$

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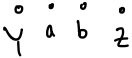
- If $\lim_{n\to\infty} M_n$ does not exist in an almost sure sense, then we must have $\mathbb{P}(Y < Z) > 0$, and hence, there must exist real numbers $0 < a < b$ such that

$$\mathbb{P}[Y < a < b < Z] > 0.$$



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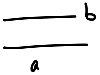

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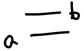
- For this to happen, it must be the case that M_n crosses from below a to above b infinitely many times with positive probability.
- To bound this probability, suppose that $M_0 \leq a$ and let τ be the first time that the martingale crosses b.



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- Note that if $\tau \leq n$, then $\widetilde{M}_n = M_\tau \geq b$, so that

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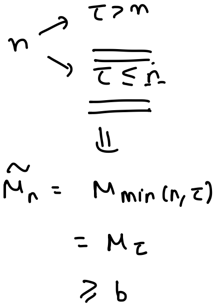

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- Hence, we get that
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and now we can take the limit on the left hand side to see that
$$\mathbb{P}[\tau < \infty] \leq a/b. < 1 \qquad (a < b)$$

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- Therefore, the probability of having k crossings is $\leq (a/b)^k$, and now we can take the limit $k \to \infty$.