STATS 217: Introduction to Stochastic Processes I

Lecture 3

The Ballot Problem

- Consider an election with two candidates A and B.
- Suppose that a votes have been cast for A and b votes have been cast for b where $a > b$.
- After the votes have been cast, they are counted in a uniformly random order.
- Since $a > b$, after all the votes are counted, A emerges as the winner.
- What is the probability that A leads B throughout the count?

The Ballot Problem

- For $0 \le i \le a + b$, let S_i denote the lead of A after i votes have been counted.
- Hence, $S_0 = 0$ and $S_{a+b} = a - b$.

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- Therefore,

$$\mathbb{P}[A \text{ leads throughout}] = \frac{N_{a+b}^{\neq 0}(0, a-b)}{N_{a+b}(0, a-b)}.$$

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$$\mathbb{P}[A \text{ leads throughout}] = \frac{N^{\neq 0}_{a+b}(0, a-b)}{N_{a+b}(0, a-b)}.$$

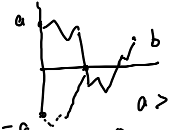
↳ we have a formula for this

$$N_n(a, b) = \binom{n}{\frac{n+b-a}{2}}$$

- So, it only remains to compute $N^{\neq 0}_{a+b}(0, a-b)$.

The Ballot Problem

We need to compute $N^{\neq 0}_{a+b}(0, a-b)$.



$a > 0,\ b > 0$

$N^{0}_{n}(a, b) = N_{n}(-a, b).$

$$N^{\neq 0}_{a+b}(0, \underbrace{a-b}_{>0}) = N^{\neq 0}_{a+b-1}(1, a-b)$$

$$= N_{a+b-1}(1, a-b) - N^{0}_{a+b-1}(1, a-b)$$

$$= N_{a+b-1}(1, a-b) - N_{a+b-1}(-1, a-b)$$

The Ballot Problem

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&= \binom{a+b-1}{a-1} - \binom{a+b-1}{a} \\
&= \frac{a-b}{a+b} \cdot N_{a+b}(0, a-b).
\end{aligned}
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Hence,

$$\mathbb{P}[A \text{ leads throughout}] = \frac{a-b}{a+b}.$$

The Ballot Problem

- One way to reinterpret the conclusion of the Ballot problem is that for any $a > b \geq 0$ and for a simple symmetric random walk starting from $S_0 = 0$,

$$\mathbb{P}[S_i > 0 \quad \forall i = 1, \ldots, a+b-1 \mid S_{a+b} = a-b] = \frac{a-b}{a+b}.$$

$$\mathbb{P}[S_1 > 0, \ldots, S_{a+b-1} > 0, S_{a+b} = a-b]$$

$$= \frac{a-b}{a+b} \cdot \mathbb{P}[S_{a+b} = a-b].$$

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- Rewritten in more convenient notation, for any integers $k, n > 0$,

$$\mathbb{P}[S_1 > 0, \dots, S_{n-1} > 0, S_n = k] = \frac{k}{n} \cdot \mathbb{P}[S_n = k]. \quad (*)$$

- On the homework, you will explore variants of this.

Path counting and applications

So far, we used path counting to answer the following questions:

- What is the distribution of the first time that a symmetric simple random walk returns to its starting point?
- What is the probability that in a uniformly random counting of the votes, the candidate with more votes stays ahead throughout?

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Today we will study the following questions:

- What is the distribution of the first time that a symmetric simple random walk hits 1?

τ_1 = first time to hit 1

$\mathbb{P}(\tau_1 < \infty) = 1$ pmf(τ_1)

$\mathbb{E}[\tau_1] = \infty$ prob$(\tau_1 = k)$

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- If the random walk is run for $2n$ steps, what is the distribution of the *last* time that the random walk visits its starting point?
- What is the distribution of the fraction of the time that the random walk is positive?...
- and more!

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- Recall that $S_j = \sum_{i=1}^{j} X_j$ where $X_1, X_2, \dots$ are i.i.d. Rademacher random variables.

$S_0 = 0$
$S_1 = X_1$
$S_2 = X_1 + X_2$ .....

each X_i is a coin flip
$+1$ w.p. $\frac{1}{2}$
-1 w.p. $\frac{1}{2}$

$$(X_1, \dots, X_n) \sim (X_n, \dots, X_1)$$
$$S_1 = X_1 \sim X_n = S_n - S_{n-1}$$
$$S_2 = X_1 + X_2 \sim X_n + X_{n-1} = S_n - S_{n-2}$$

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so that

$$(S_1, S_2, \ldots, S_n) \sim (S_n - S_{n-1}, S_n - S_{n-2}, \ldots, S_n).$$

Hitting time

- We want to compute $\mathbb{P}[\tau_b = k]$.

Hitting time

Recall that
$S_0 = 0$
$b > 0$

- We want to compute $\mathbb{P}[\tau_b = k]$.
- We have

$$\mathbb{P}[\tau_b = k] = \mathbb{P}[S_1 < b, \ldots, S_{k-1} < b, S_k = b] \quad .$$

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$$(S_1 \ldots S_k) \sim (S_k - S_{k-1}, \ldots, S_k - S_1, S_k)$$

Hitting time

$(S_1 \cdots S_k)$

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\end{aligned}
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$S_1 = X_1$ / $S_k - S_{k-1} = X_k$

↓ ballot problem

Hitting time

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$\mathbb{P}[S_k = 1]$

$k = 2\ell + 1$

$\Rightarrow$ must have ℓ steps -1

$\ell + 1$ steps $+1$

$2^{-(2\ell+1)} \binom{2\ell+1}{\ell}$

$b = 1$

$\frac{b}{k} = \frac{1}{k}$

$\mathbb{P}[\tau_1 = k] \sim \frac{1}{k^{3/2}}$

$\mathbb{E}[\tau_1] \approx \sum k \cdot \frac{1}{k^{3/2}} = \sum \frac{1}{\sqrt{k}}$ $\quad \mathbb{P}[S_k = 1] = \frac{c}{\sqrt{k}}$

Last return to 0

- Let $(S_n)_{n\geq 0}$ be a symmetric simple random walk starting from $S_0 = 0$.
- Let
$$\hat{\tau}_{2n} := \max\{0 \leq i \leq 2n : S_i = 0\}$$
denote the last time before $2n$ that the random walk visits 0.

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denote the last time before $2n$ that the random walk visits 0.
- What is the distribution of $\hat{\tau}_{2n}$?

Last return to 0

- On the homework, you will use the reflection principle to show that for any integer $n > 0$,

$$\mathbb{P}[S_1 \neq 0, \ldots, S_{2n} \neq 0 \mid S_0 = 0] = \mathbb{P}[S_{2n} = 0].$$

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- Given this, we have

$$\mathbb{P}[\hat{\tau}_{2n} = 2k] = \mathbb{P}[S_{2k} = 0, S_{2k+1} \neq 0, \ldots, S_{2n} \neq 0]$$

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$$\begin{aligned}\mathbb{P}[\hat{\tau}_{2n} = 2k] &= \mathbb{P}[S_{2k} = 0, S_{2k+1} \neq 0, \dots, S_{2n} \neq 0] \\ &= \mathbb{P}[S_{2k} = 0]\mathbb{P}[S_{2k+1} \neq 0, \dots, S_{2n} \neq 0 \mid S_{2k} = 0]\end{aligned}$$

$$= \mathbb{P}[S_1 \neq 0 \dots S_{2n-2k} \neq 0 \mid S_0 = 0]$$

$$= \mathbb{P}[S_{2n-2k} = 0].$$

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- Notice, in particular, that

$$\mathbb{P}[\hat{\tau}_{2n} = 2k] = \mathbb{P}[\hat{\tau}_{2n} = 2n - 2k](!)$$

Last return to 0

For large k, by Stirling's formula, we have

$$\mathbb{P}[S_{2k} = 0] \sim \frac{1}{\sqrt{\pi k}},$$

$\mathbb{P}[S_{2k+1} = 1] = \frac{c}{\sqrt{k}}$

so that

Last return to 0

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where

$$f(x) = \frac{1}{\pi\sqrt{x(1-x)}} \quad 0 < x < 1.$$

Last return to 0

Plot of $f(x) = \frac{1}{\pi\sqrt{x(1-x)}}$ for $0 < x < 1$.

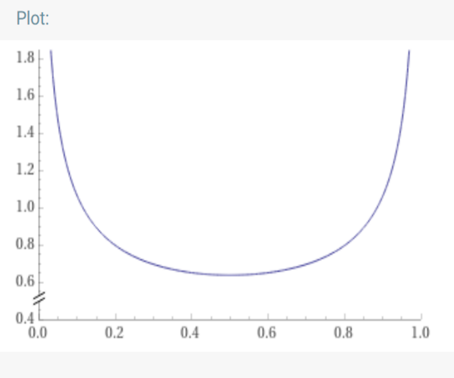

Image generated using Wolfram Alpha

The Arcsine Law

- For large k, n, we have

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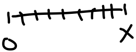
$$\sim \int_0^x f(t)dt$$

$$= \frac{2}{\pi} \cdot \arcsin \sqrt{x}.$$

$x \in (0,1)$

what is the prob that the last 0 is in first 10%

$\rightarrow \frac{2}{\pi} \cdot \arcsin \sqrt{0.1}$



The Arcsine Law

Plot of $g(x) = \frac{2}{\pi} \cdot \arcsin \sqrt{x}$ for $0 \le x \le 1$.

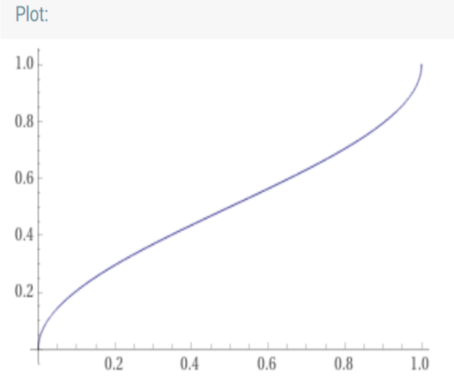


Image generated using Wolfram Alpha

The Arcsine Law

Some values of $g(x) = \frac{2}{\pi} \cdot \arcsin \sqrt{x}$. } arcsine distribution

$$x = 0.1 \quad g(x) \approx 0.204.$$
$$x = 0.2 \quad g(x) \approx 0.295.$$
$$x = 0.3 \quad g(x) \approx 0.369.$$
$$x = 0.4 \quad g(x) \approx 0.435.$$
$$x = 0.5 \quad g(x) = 0.5.$$
$$x = 0.6 \quad g(x) \approx 0.565.$$
$$x = 0.7 \quad g(x) \approx 0.631.$$
$$x = 0.8 \quad g(x) \approx 0.705.$$
$$x = 0.9 \quad g(x) \approx 0.796.$$

The Arcsine Law

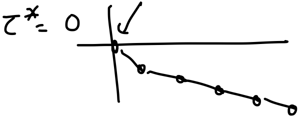
Using similar arguments, one can show that,

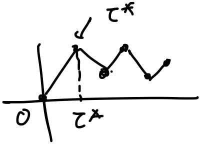
$$\mathbb{P}\left[\frac{|\{1 \le i \le n : S_i > 0\}|}{n} \le x\right] \to \frac{2}{\pi} \cdot \arcsin \sqrt{x}.$$

and also that

$$\mathbb{P}\left[\frac{t_*}{n} \le x\right] \to \frac{2}{\pi} \cdot \arcsin \sqrt{x},$$

where $0 \le t_* \le n$ is the first time when the random walk takes on its maximum value.





Some consequences of the Arc[redacted] Law

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- In particular, lead changes are most likely near the start and the end of the game. Similarly, the time of the largest lead is most likely to be near the start and the end of the game.