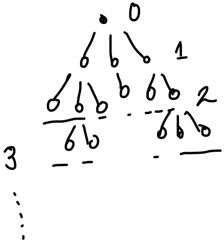
STATS 217: Introduction to Stochastic Processes I

Lecture 4

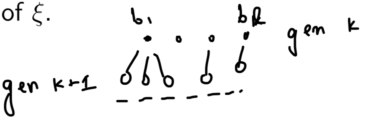
Branching processes

- Consider a single bacterium in an ideal environment. We call this the generation 0 bacterium.
- This bacterium gives birth to ξ bacteria, where ξ is a non-negative integer valued random variable. We call these the generation 1 bacteria.



Branching processes

- Consider a single bacterium in an ideal environment. We call this the generation 0 bacterium.
- This bacterium gives birth to ξ bacteria, where ξ is a non-negative integer valued random variable. We call these the generation 1 bacteria.
- Generally, let the generation k bacteria be $b_1, \ldots, b_{\cancel{k}\,\ell}$. Then, b_i gives birth to ξ_i bacteria where $\xi_1, \ldots, \xi_k$ are i.i.d. copies of ξ.



Branching processes

- Consider a single bacterium in an ideal environment. We call this the generation 0 bacterium.
- This bacterium gives birth to ξ bacteria, where ξ is a non-negative integer valued random variable. We call these the generation 1 bacteria.
- Generally, let the generation k bacteria be $b_1, \ldots, b_k$. Then, b_i gives birth to ξ_i bacteria where $\xi_1, \ldots, \xi_k$ are i.i.d. copies of ξ.
- What is the probability that the bacteria population goes extinct?

Branching processes

- Consider a single bacterium in an ideal environment. We call this the generation 0 bacterium.
- This bacterium gives birth to ξ bacteria, where ξ is a non-negative integer valued random variable. We call these the generation 1 bacteria.
- Generally, let the generation k bacteria be $b_1, \ldots, b_k$. Then, b_i gives birth to ξ_i bacteria where $\xi_1, \ldots, \xi_k$ are i.i.d. copies of ξ.
- What is the probability that the bacteria population goes extinct?
- This problem was studied by Galton and Watson in relation to the propagation of last names in Victorian England.

Branching processes

Let Z_n denote the number of bacteria in generation n and let $(\xi_{i,j})$ denote i.i.d. copies of ξ. Then,

- $Z_0 = 1$,

Branching processes

Let Z_n denote the number of bacteria in generation n and let $(\xi_{i,j})$ denote i.i.d. copies of ξ. Then,

- $Z_0 = 1$,
- $Z_1 = \xi_{0,1}$,

Branching processes

Let Z_n denote the number of bacteria in generation n and let $(\xi_{i,j})$ denote i.i.d. copies of ξ. Then,

- $Z_0 = 1$,
- $Z_1 = \xi_{0,1}$,
- $Z_2 = \sum_{i=1}^{Z_1} \xi_{1,i}$, ...

one summand for each bacterium in generation 1

Branching processes

Let Z_n denote the number of bacteria in generation n and let $(\xi_{i,j})$ denote i.i.d. copies of ξ. Then,

- $Z_0 = 1$,
- $Z_1 = \xi_{0,1}$,
- $Z_2 = \sum_{i=1}^{Z_1} \xi_{1,i}$, ...
- $Z_k = \sum_{i=1}^{Z_{k-1}} \xi_{k-1,i}$.

Note that if $Z_i = 0$ for some $i \geq 1$, then $Z_j = 0$ for all $j \geq i$. This corresponds to the extinction of the population.

Branching processes

Let Z_n denote the number of bacteria in generation n and let $(\xi_{i,j})$ denote i.i.d. copies of ξ. Then,

- $Z_0 = 1$,
- $Z_1 = \xi_{0,1}$,
- $Z_2 = \sum_{i=1}^{Z_1} \xi_{1,i}$, ...
- $Z_k = \sum_{i=1}^{Z_{k-1}} \xi_{k-1,i}$.

Note that if $Z_i = 0$ for some $i \geq 1$, then $Z_j = 0$ for all $j \geq i$. This corresponds to the extinction of the population.

Formally, we say that 0 is an **absorbing state** for the process $(Z_n)_{n \geq 0}$.

Branching processes

- We have a branching process $(Z_n)_{n\geq 0}$ with **offspring distribution** ξ.
- We are interested in the probability that the population survives i.e.

$$\mathbb{P}[Z_n \geq 1 \quad \forall n].$$

Branching processes

- We have a branching process $(Z_n)_{n\geq 0}$ with **offspring distribution** ξ.
- We are interested in the probability that the population survives i.e.

$$\mathbb{P}[Z_n \geq 1 \quad \forall n].$$

- Trivial case: Suppose $\underbrace{\mathbb{P}[\xi \geq 1] = 1}_{\text{every individual has atleast one child}}$. Then, $\mathbb{P}[Z_n \geq 1 \quad \forall n] = 1$.

Branching processes

- We have a branching process $(Z_n)_{n\geq 0}$ with **offspring distribution** ξ.
- We are interested in the probability that the population survives i.e.

$$\mathbb{P}[Z_n \geq 1 \quad \forall n].$$

- Trivial case: Suppose $\mathbb{P}[\xi \geq 1] = 1$. Then, $\mathbb{P}[Z_n \geq 1 \quad \forall n] = 1$.
- Hence, we may assume that for all integers $k \geq 0$,

$$\mathbb{P}[\xi = k] =: p_k$$

with $0 < p_0 < 1$.

Expected size of generation n

Suppose that $\mu := \mathbb{E}[\xi]$. What is the expectation of Z_n?

$$Z_n = \sum_{i=1}^{Z_{n-1}} \xi_i$$

$$Y = \sum_{i=1}^{K} \xi_i$$

$$\mathbb{E}[Y] = K\,\mathbb{E}[\xi]$$
$$= K\mu$$

Expected size of generation n

Suppose that $\mu := \mathbb{E}[\xi]$. What is the expectation of Z_n?

- $\mathbb{E}[Z_0] = 1$.

Expected size of generation n

Suppose that $\mu := \mathbb{E}[\xi]$. What is the expectation of Z_n?

- $\mathbb{E}[Z_0] = 1$.
- $\mathbb{E}[Z_1] = \mathbb{E}[\xi_{0,1}] = \mu$.

Expected size of generation n

Suppose that $\mu := \mathbb{E}[\xi]$. What is the expectation of Z_n?

- $\mathbb{E}[Z_0] = 1$.
- $\mathbb{E}[Z_1] = \mathbb{E}[\xi_{0,1}] = \mu$.
-

$$\mathbb{E}[Z_2] = \mathbb{E}\left[\sum_{i=1}^{Z_1} \xi_{1,i}\right]$$

condition on Z_1

$$= \sum_{z \geq 0} \underbrace{\mathbb{E}\left[\sum_{i=1}^{z} \xi_{1,i} \mid Z_1 = z\right]}_{z\mu} \mathbb{P}r[Z_1 = z]$$

$$= \sum_{z \geq 0} \mu \cdot \underbrace{z \mathbb{P}[Z_1 = z]}_{} = \mu \cdot \mu$$

Expected size of generation n

Suppose that $\mu := \mathbb{E}[\xi]$. What is the expectation of Z_n?

- $\mathbb{E}[Z_0] = 1$.
- $\mathbb{E}[Z_1] = \mathbb{E}[\xi_{0,1}] = \mu$.
-

$$
\begin{aligned}
\mathbb{E}[Z_2] &= \mathbb{E}\left[\sum_{i=1}^{Z_1} \xi_{1,i}\right] \\
&= \sum_{z \geq 0} \mathbb{E}\left[\sum_{i=1}^{z} \xi_{i,1}\right] \mathbb{P}[Z_1 = z] \\
&= \sum_{z \geq 0} z\mu\mathbb{P}[Z_1 = z] \\
&= \mu \sum_{z \geq 0} z\mathbb{P}[Z_1 = z] \\
&= \mu \cdot \mathbb{E}[Z_1] = \mu^2.
\end{aligned}
$$

Subcritical case

by inductive hypothesis

- Similarly, $\mathbb{E}[Z_n] = \mu\mathbb{E}[Z_{n-1}] = \mu^n$.

Subcritical case

- Similarly, $\mathbb{E}[Z_n] = \mu \mathbb{E}[Z_{n-1}] = \mu^n$.
- This shows that if $\mu < 1$, then with probability 1, the population becomes extinct.

Subcritical case

- Similarly, $\mathbb{E}[Z_n] = \mu \mathbb{E}[Z_{n-1}] = \mu^n$.
- This shows that if $\mu < 1$, then with probability 1, the population becomes extinct.
- Indeed, if $\mu < 1$ (this is called the **subcritical case**), then

$$\mathbb{P}[Z_n \geq 1] \leq \mathbb{E}[Z_n] = \mu^n \to 0.$$

Z_n is nonneg integer valued

Subcritical case

- Similarly, $\mathbb{E}[Z_n] = \mu\mathbb{E}[Z_{n-1}] = \mu^n$.
- This shows that if $\mu < 1$, then with probability 1, the population becomes extinct.
- Indeed, if $\mu < 1$ (this is called the **subcritical case**), then

$$\mathbb{P}[Z_n \geq 1] \leq \mathbb{E}[Z_n] = \mu^n \to 0.$$

- What about the case when $\mu \geq 1$?
- If $\mu = 1$, then $\mathbb{E}[Z_n] = 1$ and if $\mu > 1$, then $\mathbb{E}[Z_n] \to \infty$, but this doesn't say anything about the probability of survival.

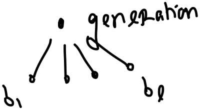
First step analysis

- To analyse the case $\mu \geq 1$, we will use first step analysis.

First step analysis

- To analyse the case $\mu \geq 1$, we will use first step analysis.
- Let ρ denote the probability that the population eventually dies out so that

$$\rho = \mathbb{P}[Z_n = 0 \text{ for some } n \geq 1].$$



note that population goes extinct $\iff$ each of the subpops. starting from b_i also go extinct.

$$\mathbb{P}[\text{subpop. of } b_1 \text{ goes extinct}] \stackrel{?}{=} \rho.$$

First step analysis

- To analyse the case $\mu \geq 1$, we will use first step analysis.
- Let ρ denote the probability that the population eventually dies out so that

$$\rho = \mathbb{P}[Z_n = 0 \text{ for some } n \geq 1].$$

- Suppose that the bacterium b in generation 0 has k children $b_1, \ldots, b_k$. Then, the population dies out if and only if the subpopulations starting at $b_1, \ldots, b_k$ die out. Moreover, the probability of each of these subpopulations dying out is also ρ.

$$\text{prob}[\text{extinct} \mid Z_1 = k] = \rho^k$$

(by independence of subpopulations)

First step analysis

- Therefore,

$$\rho = \sum_{k=0}^{\infty} \overbrace{\mathbb{P}[\xi_{0,1} = k]}^{\mathbb{P}(Z_1 = k)}\,\overbrace{\rho^k}^{\mathbb{P}(\text{extinction} \mid Z_1 = k)} = \sum_{k=0}^{\infty} p_k \rho^k = \phi(\rho),$$

where

$$\phi(z) := \sum_{k=0}^{\infty} p_k z^k = \mathbb{E}[z^X]$$

$\mathbb{P}[X = k] = p_k$

is the **generating function** of $(p_k)_{k \geq 0}$.

First step analysis

- Therefore,

$$\rho = \sum_{k=0}^{\infty} \mathbb{P}[\xi_{0,1} = k]\rho^k = \sum_{k=0}^{\infty} p_k \rho^k = \phi(\rho),$$

  where

$$\phi(z) := \sum_{k=0}^{\infty} p_k z^k$$

  is the **generating function** of $(p_k)_{k \geq 0}$.
- So, we see that the probability of extinction is a fixed point of the generating function i.e. a solution of

$$\rho = \phi(\rho).$$

First step analysis

- We saw that the probability of extinction is a solution of

$$\rho = \phi(\rho) = \sum_{k \geq 0} p_k \rho^k.$$

spoiler alert:
we can have
atmost two
solutions in [0, 1]

First step analysis

- We saw that the probability of extinction is a solution of

$$\rho = \phi(\rho) = \sum_{k \geq 0} p_k \rho^k.$$

- Since

$$\phi(1) = \sum_{k \geq 0} p_k = 1,$$

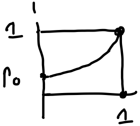
we see that 1 is always a solution of $\rho = \phi(\rho)$.

First step analysis

- We saw that the probability of extinction is a solution of

$$\rho = \phi(\rho) = \sum_{k \geq 0} p_k \rho^k.$$

we'll see that ϕ looks like



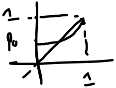
- Since

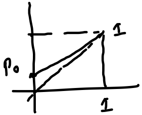
$$\phi(1) = \sum_{k \geq 0} p_k = 1,$$

we see that 1 is always a solution of $\rho = \phi(\rho)$.

- However, this does not mean that the extinction probability is 1, since there may be other solutions to $\rho = \phi(\rho)$.

2 solns





1 soln

Properties of the generating function

Recall that $\phi(z) = \sum_{k \geq 0} p_k z^k$.

- ϕ is non-decreasing on $[0, 1]$. $\because p_k \geq 0$

Properties of the generating function

Recall that $\phi(z) = \sum_{k \geq 0} p_k z^k$.

- ϕ is non-decreasing on $[0, 1]$.
- ϕ is continuous on $[0, 1]$.

Properties of the generating function

Recall that $\phi(z) = \sum_{k \geq 0} p_k z^k$.

- ϕ is non-decreasing on $[0, 1]$.
- ϕ is continuous on $[0, 1]$.
- $\phi(0) = p_0 \in (0, 1)$.

$\phi(0) = \sum_{k \geq 0} p_k 0^k$

$0^0 = \underline{1}$

$0^n = 0 \;\forall\, n > 1$

Properties of the generating function

Recall that $\phi(z) = \sum_{k \geq 0} p_k z^k$.

- ϕ is non-decreasing on $[0, 1]$.
- ϕ is continuous on $[0, 1]$.
- $\phi(0) = p_0 \in (0, 1)$.
- $\phi(1) = 1$.

Properties of the generating function

Recall that $\phi(z) = \sum_{k\geq 0} p_k z^k$.

- ϕ is non-decreasing on $[0, 1]$.
- ϕ is continuous on $[0, 1]$.
- $\phi(0) = p_0 \in (0, 1)$.
- $\phi(1) = 1$.
- $\phi'(z) = \sum_{k\geq 1} k p_k z^{k-1}$.

Properties of the generating function

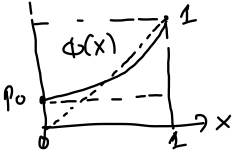
Recall that $\phi(z) = \sum_{k\geq 0} p_k z^k$.

- ϕ is non-decreasing on $[0,1]$.
- ϕ is continuous on $[0,1]$.
- $\phi(0) = p_0 \in (0,1)$.
- $\phi(1) = 1$.
- $\phi'(z) = \sum_{k\geq 1} k p_k z^{k-1}$.
- Hence, $\phi'(1) = \sum_{k\geq 1} k p_k = \mu$.

Properties of the generating function

$$= p_0 + \sum_{k\geq 1} p_k z^k$$

Recall that $\phi(z) = \sum_{k\geq 0} p_k z^k$.



- ϕ is non-decreasing on $[0,1]$.
- ϕ is continuous on $[0,1]$.
- $\phi(0) = p_0 \in (0,1)$. ← by assumption
- $\phi(1) = 1$.
- $\phi'(z) = \sum_{k\geq 1} k p_k z^{k-1}$.
- Hence, $\phi'(1) = \sum_{k\geq 1} k p_k = \mu$.
- $\phi''(z) = \sum_{k\geq 2} k(k-1) p_k z^{k-2} > 0$ for $z \in (0,1]$.

Properties of the generating function

Recall that $\phi(z) = \sum_{k \geq 0} p_k z^k$.

- ϕ is non-decreasing on $[0, 1]$.
- ϕ is continuous on $[0, 1]$.
- $\phi(0) = p_0 \in (0, 1)$.
- $\phi(1) = 1$.
- $\phi'(z) = \sum_{k \geq 1} k p_k z^{k-1}$.
- Hence, $\phi'(1) = \sum_{k \geq 1} k p_k = \mu$.
- $\phi''(z) = \sum_{k \geq 2} k(k-1) p_k z^{k-2} > 0$ for $z \in (0, 1]$.
- Hence, ϕ is strictly convex on $(0, 1]$.

Properties of the generating function

Let $g(\rho) = \phi(\rho) - \rho$. We are interested in the solutions of $g(\rho) = 0$ for $\rho = [0, 1]$.

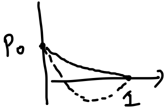
Properties of the generating function

Let $g(\rho) = \phi(\rho) - \rho$. We are interested in the solutions of $g(\rho) = 0$ for $\rho = [0, 1]$.

- We have $g(0) = p_0 \in (0, 1)$, $g(1) = 0$.

$$\phi(0) - 0 = p_0 - 0$$

$$\phi(1) - 1 = 1 - 1 = 0$$



Properties of the generating function

Let $g(\rho) = \phi(\rho) - \rho$. We are interested in the solutions of $g(\rho) = 0$ for $\rho = [0, 1]$.

- We have $g(0) = p_0 \in (0, 1)$, $g(1) = 0$.
- $g''(\rho) = \phi''(\rho) > 0$ for $\rho \in (0, 1]$.

Properties of the generating function

Let $g(\rho) = \phi(\rho) - \rho$. We are interested in the solutions of $g(\rho) = 0$ for $\rho = [0, 1]$.

- We have $g(0) = p_0 \in (0, 1)$, $g(1) = 0$.
- $g''(\rho) = \phi''(\rho) > 0$ for $\rho \in (0, 1]$.
- $g'(\rho) = \phi'(\rho) - 1$.

Properties of the generating function

Let $g(\rho) = \phi(\rho) - \rho$. We are interested in the solutions of $g(\rho) = 0$ for $\rho = [0, 1]$.

- We have $g(0) = p_0 \in (0, 1)$, $g(1) = 0$.
- $g''(\rho) = \phi''(\rho) > 0$ for $\rho \in (0, 1]$.
- $g'(\rho) = \phi'(\rho) - 1$.
- So, we have two cases:
  - If $\phi'(1) \leq 1$, then $g'(1) \leq 0$ and $g'(\rho) < 0$ for all $\rho \in [0, 1)$. Hence, the only solution of $g(\rho) = 0$ is at $\rho = 1$.

$\rightarrow$ in this, $g'(1) > 0$

$\mu \leq 1$

$g'(1) = \phi'(1) - 1$

$= \mu - 1 \leq 0$

$\Rightarrow g$ is a strictly dec fun.

$g'(1) = \phi'(1) - 1$

$= \mu - 1$

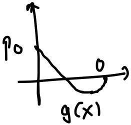
$g'(1) > 0 \Leftrightarrow \mu > 1$

Properties of the generating function

$$\phi(\rho) = \rho \iff \underbrace{\phi(\rho) - \rho}_{g(\rho)} = 0$$

Let $g(\rho) = \phi(\rho) - \rho$. We are interested in the solutions of $g(\rho) = 0$ for $\rho = [0, 1]$.

- We have $g(0) = p_0 \in (0, 1)$, $g(1) = 0$.
- $g''(\rho) = \phi''(\rho) > 0$ for $\rho \in (0, 1]$.
- $g'(\rho) = \phi'(\rho) - 1$.
- So, we have two cases:
  - If $\phi'(1) \leq 1$, then $g'(1) \leq 0$ and $g'(\rho) < 0$ for all $\rho \in [0, 1)$. Hence, the only solution of $g(\rho) = 0$ is at $\rho = 1$.
  - If $\phi'(1) > 1$, then $g'(1) > 0$. So, there exists exactly one $\rho \in (0, 1)$ such that $g(\rho) = 0$.



two cases

① $\mu = 1$: only one Root at $\rho = 1$

② $\mu > 1$: two solutions

Critical case

- We know that the extinction probability ρ is a solution of $\phi(\rho) = \rho$.

subcrit case: extinction prob = $\underline{1}$

critical case: extinction prob = $\underline{1}$?
($\mu = 1$)

= unique soln of $\phi(x) = x$ in the crit case

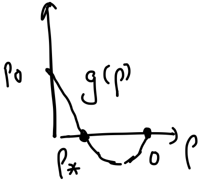
Critical case

- We know that the extinction probability ρ is a solution of $\phi(\rho) = \rho$.
- We also saw that when $\mu = \phi'(1) = 1$, this equation has only one solution: $\rho = 1$.

Critical case

- We know that the extinction probability ρ is a solution of $\phi(\rho) = \rho$.
- We also saw that when $\mu = \phi'(1) = 1$, this equation has only one solution: $\rho = 1$.
- Therefore, if $\mu = 1$ (this is called the **critical case**), we see that $\rho = 1$.

- It remains to deal with the case when $\mu > 1$ (this is called the **supercritical case**).



in this case

$$\mathbb{P}[\text{extinction}] = \begin{cases} \rho_* \\ \text{OR } 1. \end{cases}$$

3 possibilities

① ρ_*

~~②~~ 1

~~③~~ depends on additional info about ξ

will see that this is the extinction prob.

Supercritical case

- It remains to deal with the case when $\mu > 1$ (this is called the **supercritical case**).
- In this case, $\phi(\rho) = \rho$ has two solutions: $\rho^* < 1$ and 1.
- We claim that the extinction probability in this case is ρ^*.

$$\rho_n = \mathbb{P}[Z_i = 0 \text{ for some } i \le n]$$

by first step analysis $= \sum_{k \ge 0} \rho_{n-1}^k p_k = \phi(\rho_{n-1})$

exactly the argument used for $\rho = \phi(\rho)$

Supercritical case

- It remains to deal with the case when $\mu > 1$ (this is called the **supercritical case**).
- In this case, $\phi(\rho) = \rho$ has two solutions: $\rho^* < 1$ and 1.
- We claim that the extinction probability in this case is ρ^*.
- To see this, let

$$\rho_n = \mathbb{P}[Z_n = 0].$$

Supercritical case

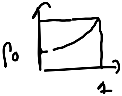
- It remains to deal with the case when $\mu > 1$ (this is called the **supercritical case**).
- In this case, $\phi(\rho) = \rho$ has two solutions: $\rho^* < 1$ and 1.
- We claim that the extinction probability in this case is ρ^*.
- To see this, let

$$\rho_n = \mathbb{P}[Z_n = 0].$$

- Then, by first step analysis, we have

$$\rho_n = \sum_{k \geq 0} p_k \rho_{n-1}^k = \phi(\rho_{n-1}).$$

Supercritical case



- We have $\rho_n = \phi(\rho_{n-1})$.
- Since ϕ is a non-decreasing function, $\rho_0 \le \rho_1 \le \rho_2 \le \ldots$

$$\underbrace{\rho}_{\text{prob of extinction.}} = \lim_{n\to\infty} \underbrace{\rho_n}_{\mathbb{P}[\text{extinction in} \le n \text{ generations}].}$$

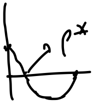
Supercritical case

$$\rho_0 = \mathbb{P}[Z_0 = 0] = 0$$
$$(\text{b/c } Z_0 = 1)$$

- We have $\rho_n = \phi(\rho_{n-1})$.
- Since ϕ is a non-decreasing function, $\rho_0 \le \rho_1 \le \rho_2 \le \dots$.
- Since $\rho_0 \le \rho^*$, it follows that

soln of $\phi(\rho) = \rho$ and $\rho^* \in (0,1)$

$$\rho_1 = \phi(\rho_0) \le \phi(\rho^*) = \rho^*.$$



$$\rho_0 \le \rho^*$$
$$\Rightarrow \underbrace{\phi(\rho_0)}_{=\rho_1} \le \phi(\rho^*) = \rho^*$$

Supercritical case

- We have $\rho_n = \phi(\rho_{n-1})$.
- Since ϕ is a non-decreasing function, $\rho_0 \leq \rho_1 \leq \rho_2 \leq ....$
- Since $\rho_0 \leq \rho^*$, it follows that

$$\rho_1 = \phi(\rho_0) \leq \phi(\rho^*) = \rho^*.$$

- Iterating this shows that $\rho_n \leq \rho^*$ for all n.

(1) we have to decide if ρ is ρ^* OR 1

(2) if we can show $\rho \leq \rho^*$, then we will be done.

(3) if we can show $\rho_n \leq \rho^* \; \forall \; n$, then will be done.

(4) $0 = \rho_0 \leq \rho^*$ (know this) now keep ϕ on both sides.

Supercritical case

- We have $\rho_n = \phi(\rho_{n-1})$.
- Since ϕ is a non-decreasing function, $\rho_0 \leq \rho_1 \leq \rho_2 \leq \ldots$.
- Since $\rho_0 \leq \rho^*$, it follows that

$$\rho_1 = \phi(\rho_0) \leq \phi(\rho^*) = \rho^*.$$

- Iterating this shows that $\rho_n \leq \rho^*$ for all n.
- Therefore,

$$\rho = \lim_{n\to\infty} \mathbb{P}[Z_n = 0] = \lim_{n\to\infty} p_n \leq \rho^*.$$

Supercritical case

- We have $\rho_n = \phi(\rho_{n-1})$.
- Since ϕ is a non-decreasing function, $\rho_0 \leq \rho_1 \leq \rho_2 \leq \ldots$.
- Since $\rho_0 \leq \rho^*$, it follows that

$$\rho_1 = \phi(\rho_0) \leq \phi(\rho^*) = \rho^*.$$

- Iterating this shows that $\rho_n \leq \rho^*$ for all n.
- Therefore,

$$\rho = \lim_{n \to \infty} \mathbb{P}[Z_n = 0] = \lim_{n \to \infty} p_n \leq \rho^*.$$

- Finally, since $\rho = \phi(\rho)$, it must be the case that $\rho = \rho^*$.

Summary

Thus, we have established the following theorem.

- Let $(Z_n)_{n\geq 0}$ be a branching process with $Z_0 = \cancel{2}\ 1$ and common offspring distribution ξ.
- Let $\mu = \mathbb{E}[\xi]$ and let $\phi(z) = \sum_{k\geq 0} \mathbb{P}[\xi = k]z^k$.
- Suppose that $0 < p_0 = \mathbb{P}[\xi = 0] < 1$.
- Let ρ be the probability of extinction.
- Then, ρ is the smallest solution of $\phi(z) = z$, $z \in [0, 1]$.
- If $\mu \leq 1$, then $\rho = 1$.
- If $\mu > 1$, then $\rho < 1$.

and in fact ρ is the smallest pos soln of $\phi(x) = x$ in $(0,]$

$\phi(0) = p_0$

assumption $p_0 = (0,1)$.

$p_0 = 0 \Rightarrow 0$ could be a soln.