STATS 217: Introduction to Stochastic Processes I

Lecture 5

The Poisson Point Process

We now begin our study of the **Poisson Point Process (PPP)** which is widely used as a (simplified) model for events such as

- time of occurrence of earthquakes,
- time of occurrence of accidents,
- starting time of telephone calls,
- time at which a new customer joins a queue at a bank,
- and many more...

The PPP will also play a crucial role in our discussion of continuous time Markov chains later in the course.

Exponential distribution

- Let $\lambda > 0$. X is said to be **exponentially distributed with rate** λ, which we will denote by $X \sim \text{Exp}(\lambda)$ if
$$\mathbb{P}(X \leq x) = 1 - e^{-\lambda x} \quad \forall x \geq 0.$$
- Equivalently, letting $f_X(x)$ denote the pdf (probability density function) of X,
$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0, \\ 0 & \text{for } x < 0. \end{cases}$$
- Observe that $\text{Exp}(\lambda) \sim \text{Exp}(1)/\lambda$. Indeed, for all $x \geq 0$,
$$\mathbb{P}(\text{Exp}(1)/\lambda \leq x) = \mathbb{P}(\text{Exp}(1) \leq \lambda x) = 1 - e^{-\lambda x} = \mathbb{P}(\text{Exp}(\lambda) \leq x).$$
- Using direct computation, one can check that for $X \sim \text{Exp}(\lambda)$
$$\mathbb{E}[X^n] = n!/\lambda^n,$$
so that
$$\mathbb{E}[\text{Exp}(\lambda)] = \lambda^{-1} \quad \text{Var}[\text{Exp}(\lambda)] = \lambda^{-2}.$$

Memorylessness of exponential distribution

Let $X \sim \text{Exp}(\lambda)$. Then, for any $t, s \geq 0$,

$$\mathbb{P}[X > t + s \mid X > t] = \mathbb{P}[X > s].$$

In words, if "waiting time" is exponentially distributed, then the probability of waiting for s more units of time doesn't depend on how long we've already waited. Indeed,

$$\mathbb{P}[X > t + s \mid X > t] = \frac{\mathbb{P}[X > t + s]}{\mathbb{P}[X > t]} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} = e^{-\lambda s} = \mathbb{P}[X > s].$$

- In fact, the exponential distribution is the **unique continuous memoryless distribution**.
- The only discrete memoryless distribution is the geometric distribution.

Sum of iid exponential random variables

Let $X_1, \ldots, X_n$ be iid Exp(1). What is the pdf of $X_1 + \cdots + X_n$?

- $n = 1$: $f_{X_1}(x) = e^{-x}$.
- $n = 2$:

$$f_{X_1+X_2}(x) = \int f_{X_1}(y) f_{X_2}(x-y)dy = \int_0^x e^{-y} e^{-(x-y)} dy = xe^{-x}.$$

- $n = 3$:

$$f_{X_1+X_2+X_3}(x) = \int f_{X_1+X_2}(y) f_{X_3}(x-y)dy = \int_0^x ye^{-x} dy = \frac{x^2}{2} e^{-x}.$$

- Similarly, by induction,

$$f_{X_1+\cdots+X_n}(x) = e^{-x} \cdot \frac{x^{n-1}}{(n-1)!}.$$

Sum of iid exponential random variables

- Let $X_1, \ldots, X_n$ be iid Exp(1). Then, $f_{X_1+\cdots+X_n}(x) = e^{-x} \cdot x^{n-1}/(n-1)!$.
- What about $X_1, \ldots, X_n$ iid Exp(λ)?
- Using $f_{S/\lambda}(x) = \lambda f_S(\lambda x)$, we get

$$f_{X_1+\cdots+X_n}(x) = \lambda e^{-\lambda x} \cdot \frac{(\lambda x)^{n-1}}{(n-1)!},$$

which is the pdf of the Gamma(n, λ) **distribution**.

Minimum of independent exponentials

Let $X_1 \sim \text{Exp}(\lambda_1), \ldots, X_n \sim \text{Exp}(\lambda_n)$ be independent. Let $X := \min(X_1, \ldots, X_n)$.

- What can we say about the distribution of X?
- $X \sim \text{Exp}(\lambda_1 + \cdots + \lambda_n)$. Indeed, for all $t \geq 0$,

$$\begin{aligned}
\mathbb{P}[X \geq t] = \mathbb{P}[X_1 \geq t, \ldots, X_n \geq t] &= \prod_{i=1}^{n} \mathbb{P}[X_i \geq t] \\
&= \prod_{i=1}^{n} e^{-\lambda_i t} = e^{-(\lambda_1 + \cdots + \lambda_n)t} \\
&= \mathbb{P}[\text{Exp}(\lambda_1 + \cdots + \lambda_n) \geq t].
\end{aligned}$$

- On the homework, you will explore $\max(X_1, \ldots, X_n)$.

Exponential races

Let $X_1 \sim \text{Exp}(\lambda_1), \ldots, X_n \sim \text{Exp}(\lambda_n)$ be independent. Let $X := \min(X_1, \ldots, X_n)$.

- What is the probability that $X = \min(X_1, \ldots, X_n) = X_1$?
- $n = 2$:

$$\mathbb{P}[X_1 = \min(X_1, X_2)] = \mathbb{P}[X_2 \geq X_1] = \int_0^\infty \mathbb{P}[X_2 \geq x] f_{X_1}(x) dx$$
$$= \int_0^\infty e^{-\lambda_2 x} \cdot \lambda_1 e^{-\lambda_1 x} dx = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

- For general n, use $X = \min(X_1, \ldots, X_n) = \min(X_1, \min(X_2, \ldots, X_n))$ and $\min(X_2, \ldots, X_n) \sim \text{Exp}(\lambda_2 + \cdots + \lambda_n)$ to get

$$\mathbb{P}[X_1 = \min(X_1, \ldots, X_n)] = \frac{\lambda_1}{\lambda_1 + \cdots + \lambda_n}.$$

Poisson distribution

- Let $\mu > 0$. X is said to have **Poisson distribution with parameter** μ, which we will denote by $X \sim \text{Pois}(\mu)$ if

$$\mathbb{P}[X = j] = e^{-\mu} \cdot \frac{\mu^j}{j!} \quad \forall j = 0, 1, 2, \ldots$$

- **Poisson approximation of the Binomial distribution**:

$$\mathbb{P}[\text{Pois}(\mu) = j] = \lim_{n \to \infty} \mathbb{P}[\text{Bin}(n, \mu/n) = j].$$

- Why? For any fixed $j = 0, 1, \ldots,$

$$\begin{aligned}\mathbb{P}[\text{Bin}(n, \mu/n) = j] &= \binom{n}{j} \left(\frac{\mu}{n}\right)^j \left(1 - \frac{\mu}{n}\right)^{n-j} \\ &= \left(\left(1 - \frac{\mu}{n}\right)^n \cdot \frac{\mu^j}{j!}\right) \cdot \frac{n!}{n^j \cdot (n-j)!} \cdot \left(1 - \frac{\mu}{n}\right)^{-j} \\ &\to e^{-\mu} \cdot \frac{\mu^j}{j!}.\end{aligned}$$

Poisson distribution

Using this intuition, we see that

- $\mathbb{E}[\mathrm{Pois}(\mu)] = \lim_{n\to\infty} \mathbb{E}[\mathrm{Bin}(n, \mu/n)] = \mu$.
- $\mathrm{Var}(\mathrm{Pois}(\mu)) = \lim_{n\to\infty} \mathrm{Var}(\mathrm{Bin}(n, \mu/n)) = \lim_{n\to\infty} n \cdot \frac{\mu}{n}\left(1 - \frac{\mu}{n}\right) = \mu$.
- For all $t \in \mathbb{R}$,

$$\begin{aligned}\mathbb{E}[e^{tX}] &= \lim_{n\to\infty} \mathbb{E}[e^{t\sum_{i=1}^{n} \mathrm{Ber}_i(\mu/n)}] = \lim_{n\to\infty} \mathbb{E}[e^{t\,\mathrm{Ber}(\mu/n)}]^n \\ &= \lim_{n\to\infty} \left(1 + \frac{\mu}{n}(e^t - 1)\right)^n \\ &= e^{\mu(e^t-1)}.\end{aligned}$$

- As practice, you should prove these results directly using the pdf of the Poisson distribution.

Sum of independent Poisson random variables

Let $X_1 \sim \text{Pois}(\lambda_1), \ldots, X_k \sim \text{Pois}(\lambda_k)$ be independent.

- How is $X_1 + \cdots + X_k$ distributed?
- From the connection between the Poisson distribution and Binomial limits, it is clear that

$$X_1 + \cdots + X_k \sim \text{Pois}(\lambda_1 + \cdots + \lambda_k).$$

- Exercise: Show this directly using the pdf of the Poisson distribution.

Connection between exponential and Poisson distributions

- Let $W_1, W_2, \ldots$ be independent $\text{Exp}(1)$ random variables. Let $W_0 = 0$.
- You should think of W_i as **waiting times** i.e. W_1 is the time you wait before the first event happens, W_2 is the time you wait between the first and second event, and so on.
- For $t \geq 0$, let

$$N(t) = \max\{i : W_1 + \cdots + W_i \leq t\}.$$

- So, $N(t)$ denotes the number of events that happen by time t.
- In particular, $N(0) = 0$.
- It turns out that for all $t \geq 0$,

$$N(t) \sim \text{Pois}(t).$$

Connection between exponential and Poisson distributions

- For all $t \geq 0$, $N(t) \sim \text{Pois}(t)$. Why?
- For any $j \geq 0$,

$$\begin{aligned}
\mathbb{P}[N(t) = j] &= \mathbb{P}[W_1 + \cdots + W_j \leq t < W_1 + \cdots + W_j + W_{j+1}] \\
&= \int_0^t f_{W_1 + \cdots + W_j}(s) \mathbb{P}[W_{j+1} > t - s] ds \\
&= \int_0^t \left(e^{-s} \cdot \frac{s^{n-1}}{(n-1)!} \right) \cdot e^{-(t-s)} ds \\
&= \frac{e^{-t}}{(n-1)!} \int_0^t s^{n-1} ds \\
&= e^{-t} \cdot \frac{t^n}{n!} \\
&= \mathbb{P}[\text{Pois}(t) = j].
\end{aligned}$$

Connection between exponential and Poisson distributions

For $t \geq 0$, let $N(t) = \max\{i : W_1 + \cdots + W_i \leq t\}$.

- We saw that $N(t) \sim \text{Pois}(t)$.
- We also have for any $0 \leq s < t$ that

$$N(t) - N(s) \text{ and } \{N(u)\}_{0 \leq u \leq s} \text{ are independent.}$$

- Why? This follows from the memorylessness property of the exponential distribution.

Connection between exponential and Poisson distributions

For any $0 \leq s < t$ that

$$N(t) - N(s) \text{ and } \{N(u)\}_{0 \leq u \leq s} \text{ are independent.}$$

- Suppose $N(s) = k$ and the **arrival times** before s are $0 \leq t_1 \leq \cdots \leq t_k \leq s$.
- This just means that $W_1 = t_1, W_1 + W_2 = t_2, \ldots, W_1 + \cdots + W_k = t_k$.
- Since $N(s) = k$, we must have $W_{k+1} \geq s - t_k$.
- But by the memorylessness property of the exponential distribution

$$\mathbb{P}[W_{k+1} > s - t_k + t \mid W_{k+1} > s - t_k] = \mathbb{P}[W_{k+1} > t] = e^{-t}.$$

- So, the waiting times for arrivals after s are iid Exp(1) random variables which are independent of $\{N(u)\}_{0 \leq u \leq s}$.