STATS 217: Introduction to Stochastic Processes I

Lecture 5

The Poisson Point Process

Chapter 2 of EOSP.

We now begin our study of the **Poisson Point Process (PPP)** which is widely used as a (simplified) model for events such as

- time of occurrence of earthquakes,
- time of occurrence of accidents,
- starting time of telephone calls,
- time at which a new customer joins a queue at a bank,
- and many more...

The PPP will also play a crucial role in our discussion of continuous time Markov chains later in the course.

Exponential distribution

- Let $\lambda > 0$. X is said to be **exponentially distributed with rate** λ, which we will denote by $X \sim \text{Exp}(\lambda)$ if

$$* \underbrace{\text{CDF}} \qquad \mathbb{P}(X \le x) = 1 - e^{-\lambda x} \quad \forall x \ge 0.$$

- Equivalently, letting $f_X(x)$ denote the pdf (probability density function) of X,

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \ge 0, \\ 0 & \text{for } x < 0. \end{cases}$$

$* \underbrace{\text{CCDF}}$

$\mathbb{P}(X \geq x)$

$= e^{-\underbrace{\lambda x}}$

$x \geq 0$

$f_X(x)\,dx$ "=" $\mathbb{P}$[lying in $[x, x+dx]$]

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$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0, \\ 0 & \text{for } x < 0. \end{cases}$$

- Observe that $\text{Exp}(\lambda) \sim \text{Exp}(1)/\lambda$. has the same distribution.

$$\mathbb{P}\left(\frac{\text{Exp}(1)}{\lambda} \leq x\right) = \mathbb{P}(\text{Exp}(1) \leq \lambda x)$$

$$= 1 - e^{-\lambda x} = \mathbb{P}(\text{Exp}(\lambda) \leq x)$$

Exponential distribution

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$$\mathbb{P}(X \leq x) = 1 - e^{-\lambda x} \quad \forall x \geq 0.$$
- Equivalently, letting $f_X(x)$ denote the pdf (probability density function) of X,
$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0, \\ 0 & \text{for } x < 0. \end{cases}$$
- Observe that $\text{Exp}(\lambda) \sim \text{Exp}(1)/\lambda$. Indeed, for all $x \geq 0$,
$$\mathbb{P}(\text{Exp}(1)/\lambda \leq x) = \mathbb{P}(\text{Exp}(1) \leq \lambda x) = 1 - e^{-\lambda x} = \mathbb{P}(\text{Exp}(\lambda) \leq x).$$

Exponential distribution

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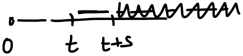
- Equivalently, letting $f_X(x)$ denote the pdf (probability density function) of X,
$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0, \\ 0 & \text{for } x < 0. \end{cases}$$
- Observe that $\text{Exp}(\lambda) \sim \text{Exp}(1)/\lambda$. Indeed, for all $x \geq 0$,
$$\mathbb{P}(\text{Exp}(1)/\lambda \leq x) = \mathbb{P}(\text{Exp}(1) \leq \lambda x) = 1 - e^{-\lambda x} = \mathbb{P}(\text{Exp}(\lambda) \leq x).$$
- Using direct computation, one can check that for $X \sim \text{Exp}(\lambda)$
$$\mathbb{E}[X^n] = n!/\lambda^n,$$
$X \sim Exp(1)$
$\mathbb{E}[X^n] = n!$
so that
$$\mathbb{E}[\text{Exp}(\lambda)] = \lambda^{-1} \quad \text{Var}[\text{Exp}(\lambda)] = \lambda^{-2}.$$

Memorylessness of exponential distribution

Let $X \sim \mathrm{Exp}(\lambda)$. Then, for any $t, s \geq 0$,



$$\mathbb{P}[X > t + s \mid X > t] = \mathbb{P}[X > s].$$

In words, if "waiting time" is exponentially distributed, then the probability of waiting for s more units of time doesn't depend on how long we've already waited.

$$\mathbb{P}[X > t+s \mid X > t] = \frac{\mathbb{P}[X > t+s \cancel{\cap X > t}]}{\mathbb{P}[X > t]}$$

$$= \frac{\exp(-\lambda(t+s))}{\exp(-\lambda t)}$$

$$= \exp(-\lambda s)$$

$$= \mathbb{P}[X > S]$$

Memorylessness of exponential distribution

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- In fact, the exponential distribution is the **unique continuous memoryless distribution**. $f(t) := \mathbb{P}[X > t]$

memorylessness $\Rightarrow \dfrac{f(t+s)}{f(t)} = f(s) \Leftrightarrow f(t+s) = f(t)f(s)$

Memorylessness of exponential distribution

$f(2) = f(1)^2$

$\vdots$

$f(n) = f(1)^n$

Let $X \sim \text{Exp}(\lambda)$. Then, for any $t, s \geq 0$,

$$\mathbb{P}[X > t + s \mid X > t] = \mathbb{P}[X > s].$$

In words, if "waiting time" is exponentially distributed, then the probability of waiting for s more units of time doesn't depend on how long we've already waited. Indeed,

$$\mathbb{P}[X > t + s \mid X > t] = \frac{\mathbb{P}[X > t + s]}{\mathbb{P}[X > t]} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} = e^{-\lambda s} = \mathbb{P}[X > s].$$

- In fact, the exponential distribution is the **unique continuous memoryless distribution**.
- The only discrete memoryless distribution is the geometric distribution.

~~iid~~ iid trials & # of ~~times~~ ✓ trials to first success

Sum of iid exponential random variables

later we will
look at
the
minimum

Let $X_1, \dots, X_n$ be iid Exp(1). What is the pdf of $X_1 + \cdots + X_n$?

* punch line: sum & minimum have nice closed forms.

* sum: gamma dis.

* min: exponential dis.

Sum of iid exponential random variables

Let $X_1, \ldots, X_n$ be iid Exp(1). What is the pdf of $X_1 + \cdots + X_n$?

- $n = 1$: $f_{X_1}(x) = e^{-x}$.

Sum of iid exponential random variables

Let $X_1, \ldots, X_n$ be iid Exp(1). What is the pdf of $X_1 + \cdots + X_n$?

- $n = 1$: $f_{X_1}(x) = e^{-x}$.
- $n = 2$:

$$f_{X_1+X_2}(x) = \int \widetilde{f_{X_1}(y)}\,\widetilde{f_{X_2}(x-y)}\,dy = \int_0^x e^{-y} e^{-(x-y)} dy = xe^{-x}.$$

continuous analog of

$$\mathbb{P}(X_1 + X_2 = x) = \sum_y \mathbb{P}(X_1 = y)\,\mathbb{P}(X_2 = x - y).$$

Sum of iid exponential random variables

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- $n = 2$:

$$f_{X_1+X_2}(x) = \int f_{X_1}(y) f_{X_2}(x-y)dy = \int_0^x e^{-y} e^{-(x-y)} dy = xe^{-x}.$$

- $n = 3$: $X_1 + X_2 + X_3 = (X_1 + X_2) + X_3$

$$f_{X_1+X_2+X_3}(x) = \int f_{X_1+X_2}(y) f_{X_3}(x-y)dy = \int_0^x ye^{-x} dy = \frac{x^2}{2} e^{-x}.$$

.

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- Similarly, by induction,

$$f_{X_1+\cdots+X_n}(x) = e^{-x} \cdot \frac{x^{n-1}}{(n-1)!}.$$

Sum of iid exponential random variables

- Let $X_1, \ldots, X_n$ be iid Exp(1). Then, $f_{X_1+\cdots+X_n}(x) = e^{-x} \cdot x^{n-1}/(n-1)!$.

Sum of iid exponential random variables

- Let $X_1, \ldots, X_n$ be iid Exp(1). Then, $f_{X_1+\cdots+X_n}(x) = e^{-x} \cdot x^{n-1}/(n-1)!$.
- What about $X_1, \ldots, X_n$ iid Exp(λ)?

Sum of iid exponential random variables

- Let $X_1, \ldots, X_n$ be iid Exp(1). Then, $f_{X_1+\cdots+X_n}(x) = e^{-x} \cdot x^{n-1}/(n-1)!$.
- What about $X_1, \ldots, X_n$ iid Exp(λ)? $X_1 \sim Y_1/\lambda \quad Y_1 \sim \text{Exp}(1)$
- Using $f_{S/\lambda}(x) = \lambda f_S(\lambda x)$, $X_1 + \cdots + X_n \sim \dfrac{Y_1 + \cdots + Y_n}{\lambda}$

Sum of iid exponential random variables

- Let $X_1, \ldots, X_n$ be iid Exp(1). Then, $f_{X_1+\cdots+X_n}(x) = e^{-x} \cdot x^{n-1}/(n-1)!$.
- What about $X_1, \ldots, X_n$ iid Exp(λ)?
- Using $f_{S/\lambda}(x) = \lambda f_S(\lambda x)$, we get

$$f_{X_1+\cdots+X_n}(x) = \lambda e^{-\lambda x} \cdot \frac{(\lambda x)^{n-1}}{(n-1)!},$$

which is the pdf of the Gamma(n, λ) **distribution**.

Minimum of independent exponentials

Let $X_1 \sim \text{Exp}(\lambda_1), \ldots, X_n \sim \text{Exp}(\lambda_n)$ be independent. Let $X := \min(X_1, \ldots, X_n)$.

$$X \sim \text{Exp}(\lambda_1 + \cdots + \lambda_n).$$

Minimum of independent exponentials

Let $X_1 \sim \text{Exp}(\lambda_1), \ldots, X_n \sim \text{Exp}(\lambda_n)$ be independent. Let $X := \min(X_1, \ldots, X_n)$.

- What can we say about the distribution of X?

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- $X \sim \mathrm{Exp}(\lambda_1 + \cdots + \lambda_n)$. Indeed, for all $t \geq 0$,

$$\mathbb{P}[X \geq t] = \mathbb{P}[X_1 \geq t, \dots, X_n \geq t] = \prod_{i=1}^{n} \mathbb{P}[X_i \geq t]$$

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$$\begin{aligned} \mathbb{P}[X \geq t] = \mathbb{P}[X_1 \geq t, \ldots, X_n \geq t] &= \prod_{i=1}^{n} \mathbb{P}[X_i \geq t] \\ &= \prod_{i=1}^{n} e^{-\lambda_i t} = e^{-(\lambda_1 + \cdots + \lambda_n)t} \\ &= \mathbb{P}[\text{Exp}(\lambda_1 + \cdots + \lambda_n) \geq t]. \end{aligned}$$

Minimum of independent exponentials

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- On the homework, you will explore $\max(X_1, \ldots, X_n)$.

Exponential races

Let $X_1 \sim \text{Exp}(\lambda_1), \ldots, X_n \sim \text{Exp}(\lambda_n)$ be independent. Let $X := \min(X_1, \ldots, X_n)$.

- What is the probability that $X = \min(X_1, \ldots, X_n) = X_1$?

$$= \frac{\lambda_1}{\lambda_1 + \cdots + \lambda_n}$$

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- What is the probability that $X = \min(X_1, \ldots, X_n) = X_1$?
- $n = 2$:

$$\mathbb{P}[X_1 = \min(X_1, X_2)] = \mathbb{P}[X_2 \geq X_1] \overset{\text{conditioning on } X_1}{=} \int_0^\infty \underbrace{\mathbb{P}[X_2 \geq x]}_{e^{-\lambda_2 x}} \underbrace{\overline{f_{X_1}(x)}}_{\substack{\text{analog of} \\ \mathbb{P}[X_1 = x]}} dx$$

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$$= \int_0^\infty e^{-\lambda_2 x} \cdot \lambda_1 e^{-\lambda_1 x} dx = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

- general n? $\frac{\lambda_1}{\lambda_1 + \cdots + \lambda_n}$

$\min(X_1, X_2, \ldots X_n)$

$= \min(X_1, \underbrace{\min(X_2, \ldots X_n)}_{\overset{\shortparallel}{\text{exponential}} \text{ w.p. } \lambda_2 + \cdots + \lambda_n})$

and now use the case $n = 2$.

Exponential races

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- $n = 2$:

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- For general n, use $X = \min(X_1, \ldots, X_n) = \min(X_1, \min(X_2, \ldots, X_n))$ and $\min(X_2, \ldots, X_n) \sim \text{Exp}(\lambda_2 + \cdots + \lambda_n)$

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Let $X_1 \sim \text{Exp}(\lambda_1), \ldots, X_n \sim \text{Exp}(\lambda_n)$ be independent. Let $X := \min(X_1, \ldots, X_n)$.

- What is the probability that $X = \min(X_1, \ldots, X_n) = X_1$?
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$$\mathbb{P}[X_1 = \min(X_1, X_2)] = \mathbb{P}[X_2 \geq X_1] = \int_0^\infty \mathbb{P}[X_2 \geq x] f_{X_1}(x) dx$$
$$= \int_0^\infty e^{-\lambda_2 x} \cdot \lambda_1 e^{-\lambda_1 x} dx = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

- For general n, use $X = \min(X_1, \ldots, X_n) = \min(X_1, \min(X_2, \ldots, X_n))$ and $\min(X_2, \ldots, X_n) \sim \text{Exp}(\lambda_2 + \cdots + \lambda_n)$ to get

$$\mathbb{P}[X_1 = \min(X_1, \ldots, X_n)] = \frac{\lambda_1}{\lambda_1 + \cdots + \lambda_n}.$$

Poisson distribution

- Let $\mu > 0$. X is said to have **Poisson distribution with parameter** μ, which we will denote by $X \sim \text{Pois}(\mu)$ if

$$\mathbb{P}[X = j] = e^{-\mu} \cdot \frac{\mu^j}{j!} \quad \forall j = 0, 1, 2, \ldots$$

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- **Poisson approximation of the Binomial distribution**:

$$\mathbb{P}[\text{Pois}(\mu) = j] = \lim_{n\to\infty} \mathbb{P}[\text{Bin}(n, \mu/n) = j].$$

think of μ as fixed, say 1.

μ is fixed, n is large, j is fixed

$$\mathbb{P}\left[\text{Bin}\left(n, \frac{\mu}{n}\right) = j\right] = \binom{n}{j}\left(\frac{\mu}{n}\right)^j\left(1 - \frac{\mu}{n}\right)^{n-j}$$

$$\xrightarrow{n\to\infty} e^{-\mu} \cdot \frac{\mu^j}{j!} = \mathbb{P}[\text{Pois}(\mu) = j].$$

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- Why? For any fixed $j = 0, 1, \ldots,$

$$\mathbb{P}[\text{Bin}(n, \mu/n) = j] = \binom{n}{j} \left(\frac{\mu}{n}\right)^j \left(1 - \frac{\mu}{n}\right)^{n-j}$$

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- Why? For any fixed $j = 0, 1, \ldots,$

$$\frac{n!}{(n-j)!} = n(n-1) \cdots (n-j+1)$$

$$\mathbb{P}[\text{Bin}(n, \mu/n) = j] = \binom{n}{j} \left(\frac{\mu}{n}\right)^j \left(1 - \frac{\mu}{n}\right)^{n-j}$$

$$\lim_{n \to \infty} \left(1 - \frac{a}{n}\right)^n = e^{-a}$$

$$= \left(\underset{\downarrow \; e^{-\mu}}{\left(1 - \frac{\mu}{n}\right)^n} \cdot \underset{=.}{\frac{\mu^j}{j!}}\right) \cdot \underbrace{\frac{n!}{n^j \cdot (n-j)!}}_{\downarrow \; 1} \cdot \underbrace{\left(1 - \frac{\mu}{n}\right)^{-j}}_{\approx e^{+j/n} \to 1}$$

if x is very small $1 + x \approx e^x$

Poisson distribution

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$$\begin{aligned}\mathbb{P}[\text{Bin}(n, \mu/n) = j] &= \binom{n}{j} \left(\frac{\mu}{n}\right)^j \left(1 - \frac{\mu}{n}\right)^{n-j} \\ &= \left(\left(1 - \frac{\mu}{n}\right)^n \cdot \frac{\mu^j}{j!}\right) \cdot \frac{n!}{n^j \cdot (n-j)!} \cdot \left(1 - \frac{\mu}{n}\right)^{-j} \\ &\to e^{-\mu} \cdot \frac{\mu^j}{j!}.\end{aligned}$$

Poisson distribution

$\mathbb{E}[Bin(n,p)]$

$= np$

Using this intuition, we see that

- $\mathbb{E}[\text{Pois}(\mu)] = \lim_{n\to\infty} \mathbb{E}[\text{Bin}(n, \mu/n)] = \mu$.

$$* \; Var(Pois(\mu)) \overset{?}{=} \lim_{n\to\infty} Var(Bin(n, \mu/n))$$

$$= \lim_{n} n \, Var\left(Ber\left(\frac{\mu}{n}\right)\right)$$

$$= n \cdot \frac{\mu}{n}\left(1-\frac{\mu}{n}\right) \rightarrow \mu .$$

Poisson distribution

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- $\mathbb{E}[\text{Pois}(\mu)] = \lim_{n\to\infty} \mathbb{E}[\text{Bin}(n, \mu/n)] = \mu$.
- $\text{Var}(\text{Pois}(\mu)) = \lim_{n\to\infty} \text{Var}(\text{Bin}(n, \mu/n)) = \lim_{n\to\infty} n \cdot \frac{\mu}{n} \left(1 - \frac{\mu}{n}\right) = \mu$.

Poisson distribution

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- $\mathbb{E}[\text{Pois}(\mu)] = \lim_{n\to\infty} \mathbb{E}[\text{Bin}(n, \mu/n)] = \mu$.
- $\text{Var}(\text{Pois}(\mu)) = \lim_{n\to\infty} \text{Var}(\text{Bin}(n, \mu/n)) = \lim_{n\to\infty} n \cdot \frac{\mu}{n}\left(1 - \frac{\mu}{n}\right) = \mu$.
- For all $t \in \mathbb{R}$, $X \sim \text{Pois}(\mu)$

$$\mathbb{E}[e^{tX}] = \lim_{n\to\infty} \mathbb{E}[e^{t\sum_{i=1}^{n} \text{Ber}_i(\mu/n)}] = \lim_{n\to\infty} \mathbb{E}[e^{t\,\text{Ber}(\mu/n)}]^n$$

Poisson distribution

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- $\mathbb{E}[\text{Pois}(\mu)] = \lim_{n\to\infty} \mathbb{E}[\text{Bin}(n, \mu/n)] = \mu.$
- $\text{Var}(\text{Pois}(\mu)) = \lim_{n\to\infty} \text{Var}(\text{Bin}(n, \mu/n)) = \lim_{n\to\infty} n \cdot \frac{\mu}{n}\left(1 - \frac{\mu}{n}\right) = \mu.$
- For all $t \in \mathbb{R}$,

$$\begin{aligned}\mathbb{E}[e^{tX}] &= \lim_{n\to\infty} \mathbb{E}[e^{t\sum_{i=1}^{n} \text{Ber}_i(\mu/n)}] = \lim_{n\to\infty} \mathbb{E}[e^{t\,\text{Ber}(\mu/n)}]^n \\ &= \lim_{n\to\infty} \left(1 + \frac{\mu}{n}(e^t - 1)\right)^n\end{aligned}$$

Poisson distribution

Using this intuition, we see that

- $\mathbb{E}[\text{Pois}(\mu)] = \lim_{n\to\infty} \mathbb{E}[\text{Bin}(n, \mu/n)] = \mu$.
- $\text{Var}(\text{Pois}(\mu)) = \lim_{n\to\infty} \text{Var}(\text{Bin}(n, \mu/n)) = \lim_{n\to\infty} n \cdot \frac{\mu}{n}\left(1 - \frac{\mu}{n}\right) = \mu$.
- For all $t \in \mathbb{R}$,

$$\begin{aligned}\mathbb{E}[e^{tX}] &= \lim_{n\to\infty} \mathbb{E}[e^{t\sum_{i=1}^{n} \text{Ber}_i(\mu/n)}] = \lim_{n\to\infty} \mathbb{E}[e^{t\,\text{Ber}(\mu/n)}]^n \\ &= \lim_{n\to\infty} \left(1 + \frac{\mu}{n}(e^t - 1)\right)^n \\ &= e^{\mu(e^t - 1)}.\end{aligned}$$

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- As practice, you should prove these results directly using the pdf of the Poisson distribution.

Sum of independent Poisson random variables

Let $X_1 \sim \text{Pois}(\lambda_1), \ldots, X_k \sim \text{Pois}(\lambda_k)$ be independent.

- How is $X_1 + \cdots + X_k$ distributed?

* you can check formally.

$$\mathrm{Poi}(\lambda_1) \approx \mathrm{Bin}\left(n, \frac{\lambda_1}{n}\right) \sim \underbrace{\mathrm{Ber}\left(\frac{\lambda_1}{n}\right) + \dots + \mathrm{Ber}\left(\frac{\lambda_1}{n}\right)}_{n \text{ ind. copies}}$$

$$\mathrm{Poi}(\lambda_1) + \dots + \mathrm{Poi}(\lambda_k)$$

$$\approx \left\{ \underbrace{\mathrm{Ber}\left(\frac{\lambda_1}{n}\right) + \dots + \mathrm{Ber}\left(\frac{\lambda_k}{n}\right)}_{Y} \right\} + \dots +$$

repeated n ind. times.

$\mathbb{P}[Y = 0] =$

$\mathbb{P}[Y = 1] =$ order $1/n$

$\mathbb{P}[Y = 2]$: order $\frac{1}{n^2}$

Sum of independent Poisson random variables

$\mathbb{P}[Y=k] = \text{order } \frac{1}{n^k}$

Let $X_1 \sim \text{Pois}(\lambda_1), \ldots, X_k \sim \text{Pois}(\lambda_k)$ be independent.

- How is $X_1 + \cdots + X_k$ distributed?
- From the connection between the Poisson distribution and Binomial limits, it is clear that

$$X_1 + \cdots + X_k \sim \text{Pois}(\lambda_1 + \cdots + \lambda_k).$$

$Y = \text{Ber}\left(\frac{\lambda_1}{n}\right) + \cdots + \text{Ber}\left(\frac{\lambda_k}{n}\right)$ $\quad Y \approx \text{Ber}\left(\frac{\lambda_1 + \cdots + \lambda_k}{n}\right)$

$\mathbb{P}[Y=1] \approx \frac{\lambda_1}{n} + \cdots + \frac{\lambda_k}{n}$

Sum of independent Poisson random variables

$$Y = \text{Ber}\left(\frac{\lambda_1}{n}\right) + \text{Ber}\left(\frac{\lambda_2}{n}\right)$$

$$1 - \left(\frac{\lambda_1+\lambda_2}{n}\right) + \frac{\lambda_1\lambda_2}{n^2}$$

$$Y=0 : \left(1-\frac{\lambda_1}{n}\right)\left(1-\frac{\lambda_2}{n}\right)$$

$$Y=1 : \left(1-\frac{\lambda_1}{n}\right)\frac{\lambda_2}{n} + \frac{\lambda_1}{n}\left(1-\frac{\lambda_2}{n}\right)$$

$$Y=2 : \frac{\lambda_1}{n} \frac{\lambda_2}{n}$$

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- How is $X_1 + \cdots + X_k$ distributed?
- From the connection between the Poisson distribution and Binomial limits, it is clear that
$$X_1 + \cdots + X_k \sim \text{Pois}(\lambda_1 + \cdots + \lambda_k).$$
- Exercise: Show this directly using the pdf of the Poisson distribution.

Connection between exponential and Poisson distributions

- Let $W_1, W_2, \dots$ be independent Exp(1) random variables. Let $W_0 = 0$.

$N(t)$ = # of arrivals by time t

$N(t) \sim \text{Pois}(t)$

We stopped here.

Connection between exponential and Poisson distributions

- Let $W_1, W_2, \ldots$ be independent Exp(1) random variables. Let $W_0 = 0$.
- You should think of W_i as **waiting times** i.e. W_1 is the time you wait before the first event happens, W_2 is the time you wait between the first and second event, and so on.

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- For $t \geq 0$, let

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- It turns out that for all $t \geq 0$,

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&= \frac{e^{-t}}{(n-1)!} \int_0^t s^{n-1} ds \\
&= e^{-t} \cdot \frac{t^n}{n!} \\
&= \mathbb{P}[\text{Pois}(t) = j].
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For $t \geq 0$, let $N(t) = \max\{i : W_1 + \cdots + W_i \leq t\}$.

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- We also have for any $0 \leq s < t$ that

$$N(t) - N(s) \text{ and } \{N(u)\}_{0 \leq u \leq s} \text{ are independent.}$$

- Why?

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- Why? This follows from the memorylessness property of the exponential distribution.

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- Since $N(s) = k$, we must have $W_{k+1} \geq s - t_k$.

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- But by the memorylessness property of the exponential distribution

$$\mathbb{P}[W_{k+1} > s - t_k + t \mid W_{k+1} > s - t_k] = \mathbb{P}[W_{k+1} > t] = e^{-t}.$$

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- So, the waiting times for arrivals after s are iid Exp(1) random variables which are independent of $\{N(u)\}_{0 \leq u \leq s}$.