STATS 217: Introduction to Stochastic Processes I

Lecture 6

Connection between exponential and Poisson distributions

- Let $W_1, W_2, \ldots$ be independent Exp(1) random variables. Let $W_0 = 0$.
- You should think of W_i as **waiting times** i.e. W_1 is the time you wait before the first event happens, W_2 is the time you wait between the first and second event, and so on.
- For $t \geq 0$, let

$$N(t) = \max\{i : W_1 + \cdots + W_i \leq t\}.$$

- So, $N(t)$ denotes the number of events that happen by time t.
- In particular, $N(0) = 0$.
- It turns out that for all $t \geq 0$,

$$N(t) \sim \text{Pois}(t).$$

Connection between exponential and Poisson distributions

- For all $t \geq 0$, $N(t) \sim \text{Pois}(t)$. Why?
- For any $j \geq 0$,

$$\begin{aligned}
\mathbb{P}[N(t) = j] &= \mathbb{P}[W_1 + \cdots + W_j \leq t < W_1 + \cdots + W_j + W_{j+1}] \\
&= \int_0^t f_{W_1+\cdots+W_j}(s)\mathbb{P}[W_{j+1} > t - s]ds \\
&= \int_0^t \left(e^{-s} \cdot \frac{s^{n-1}}{(n-1)!}\right) \cdot e^{-(t-s)}ds \\
&= \frac{e^{-t}}{(n-1)!} \int_0^t s^{n-1}ds \\
&= e^{-t} \cdot \frac{t^n}{n!} \\
&= \mathbb{P}[\text{Pois}(t) = j].
\end{aligned}$$

Connection between exponential and Poisson distributions

For $t \geq 0$, let $N(t) = \max\{i : W_1 + \cdots + W_i \leq t\}$.

- We saw that $N(t) \sim \text{Pois}(t)$.
- We also have for any $0 \leq s < t$ that

$$N(t) - N(s) \text{ and } \{N(u)\}_{0 \leq u \leq s} \text{ are independent.}$$

- Why? This follows from the memorylessness property of the exponential distribution.

Connection between exponential and Poisson distributions

For any $0 \leq s < t$ that

$$N(t) - N(s) \text{ and } \{N(u)\}_{0 \leq u \leq s} \text{ are independent.}$$

- Suppose $N(s) = k$ and the **arrival times** before s are $0 \leq \alpha_1 \leq \cdots \leq \alpha_k \leq s$.
- This just means that $W_1 = \alpha_1, W_1 + W_2 = \alpha_2, \ldots, W_1 + \cdots + W_k = \alpha_k$.
- Since $N(s) = k$, we must have $W_{k+1} \geq s - \alpha_k$.
- But by the memorylessness property of the exponential distribution

$$\mathbb{P}[W_{k+1} > s - \alpha_k + t \mid W_{k+1} > s - \alpha_k] = \mathbb{P}[W_{k+1} > t] = e^{-t}.$$

- So, the waiting times for arrivals after s are iid Exp(1) random variables which are independent of $\{N(u)\}_{0 \leq u \leq s}$.

The Poisson Point Process

Let $\lambda > 0$. A collection of random variables $\{N(s), s \geq 0\}$ is said to be a **Poisson point process with rate** λ if

P1 $N(0) = 0$,

P2 $N(t+s) - N(s) \sim \text{Pois}(\lambda t)$,

P3 $N(t)$ has independent increments, i.e., for any $t_0 < t_1 < \cdots < t_n$,

$$N(t_1) - N(t_0), N(t_2) - N(t_1), \ldots, N(t_n) - N(t_{n-1}) \text{ are independent.}$$

We already saw above that taking $W_0 = 0$, $W_1, W_2, \ldots$ to be iid $\text{Exp}(\lambda)$ and

$$N(s) := \max\{i : W_1 + \cdots + W_i \leq s\}$$

gives a Poisson process with rate λ.

Uniqueness of the construction

In fact, our construction of the PPP is unique.

- Let $\{N(s)\}_{s \geq 0}$ be a Poisson point process with rate λ.
- Let $\alpha_0 = 0$.
- For $i \geq 1$, let $\alpha_i := \inf\{t : N(t) = i\}$.
- So, α_i is the (random) i^{th} arrival time i.e. the time that the i^{th} event happens.
- Let $W_i = \alpha_i - \alpha_{i-1}$ denote the (random) waiting time for the i^{th} event.
- Then, $W_1, W_2, \ldots,$ are iid $\text{Exp}(\lambda)$.

Uniqueness of the construction

Here's the idea.

- Let's look at $W_{i+1} = \alpha_{i+1} - \alpha_i$.
- By conditioning on α_i, we have

$$\mathbb{P}[W_{i+1} > t] = \int_0^\infty \mathbb{P}[\alpha_{i+1} - s > t \mid \alpha_i = s] f_{\alpha_i}(s) ds.$$

- Note that

$$\alpha_{i+1} - s > t \mid \alpha_i = s \iff N(s+t) - N(s) = 0 \mid N(s) = i.$$

- But by the independent increment property,

$$\begin{aligned}\mathbb{P}[N(s+t) - N(s) = 0 \mid N(s) = i] &= \mathbb{P}[N(s+t) - N(s) = 0] \\ &= \mathbb{P}[\text{Pois}(\lambda t) = 0] \\ &= e^{-\lambda t}\end{aligned}$$

Uniqueness of the construction

- This shows that the waiting times $W_1, W_2, \dots$ have $\text{Exp}(\lambda)$ distribution,
- As for independence, note that we actually showed that for all s,

$$\mathbb{P}[W_{i+1} > t \mid \alpha_i = s] = e^{-\lambda t}.$$

- Since by the independent increments property,

$$\mathbb{P}[W_{i+1} > t \mid \alpha_i = s] = \mathbb{P}[W_{i+1} > t \mid \alpha_i = s, \alpha_{i-1} = *, \dots, \alpha_1 = *],$$

  this shows that $W_{i+1} = \alpha_{i+1} - \alpha_i$ is independent of $\alpha_1, \dots, \alpha_i$, and hence of $W_1, \dots, W_i$.

Infinitesimal description of the PPP

Here is another equivalent (although a bit informal) "infinitesimal" description of the Poisson point process with rate λ.

- $N(t)$ is the number of points in $[0, t]$.
- $\mathbb{P}[\text{there is a point in } [t, t + dt]] = \lambda \cdot dt$.
- The number of points in disjoint intervals are independent.

Idea: the second and third conditions, together with the Poisson approximation of the Binomial distribution show that indeed, for any $0 \leq s \leq t$,

$$N(t) - N(s) \sim \text{Pois}(\lambda(t - s)),$$

and the third condition guarantees independence of increments.

The Inhomogeneous PPP

- In many situations, the condition

$$\mathbb{P}[\text{there is a point in } [t, t+dt]] = \lambda \cdot dt$$

  is unrealistic.

- For instance, more phone calls start during the day than in the middle of the night.

- In such cases, one can consider the more general condition

$$\mathbb{P}[\text{there is a point in } [t, t+dt]] = \lambda(t) \cdot dt.$$

The Inhomogeneous PPP

The inhomogeneous PPP may be described infinitesimally by

- $N(t)$ is the number of points in $[0, t]$.
- $\mathbb{P}[\text{there is a point in } [t, t+dt]] = \lambda(t)dt$.
- The number of points in disjoint intervals are independent.

By the Poisson approximation of the Binomial distribution, for any $0 \leq s \leq t$,

$$N(t) - N(s) \sim \text{Pois}\left(\int_s^t \lambda(u)du\right).$$

Why?

$$\begin{aligned}\mathbb{P}[N(t) - N(s) = k] &\approx \mathbb{P}\left[\text{Ber}\,\frac{\lambda(s)}{n} + \text{Ber}\,\frac{\lambda(s+n^{-1})}{n} + \cdots + \text{Ber}\,\frac{\lambda(t-n^{-1})}{n} = k\right] \\ &\approx \mathbb{P}\left[\text{Pois}\left(\frac{\lambda(s)}{n} + \frac{\lambda(s+n^{-1})}{n} + \frac{\lambda(t-n^{-1})}{n}\right) = k\right] \\ &\approx \mathbb{P}\left[\text{Pois}\left(\int_s^t \lambda(u)du\right) = k\right].\end{aligned}$$

Construction of inhomogeneous PPP

Let

$$\Lambda(s,t) := \int_s^t \lambda(u)du.$$

- $N(t) = N^{\text{hom}}(\Lambda(0,t))$ is an inhomogeneous PPP, where $N^{\text{hom}}(\cdot)$ is a (homogeneous) PPP with rate 1. Why?

  Clearly $N(0) = 0$ and $N(t)$ has independent increments. As for the distribution, note that

$$\begin{aligned} N(t+dt) - N(t) &= N^{\text{hom}}(\Lambda(0,t+dt)) - N^{\text{hom}}(\Lambda(0,t)) \\ &\approx N^{\text{hom}}(\Lambda(0,t) + \lambda(t)dt) - N^{\text{hom}}(\Lambda(0,t)) \\ &\approx \lambda(t)dt. \end{aligned}$$