STATS 217: Introduction to Stochastic Processes I

Lecture 6

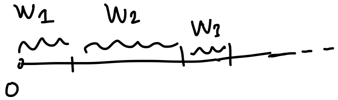
Connection between exponential and Poisson distributions

- Let $W_1, W_2, \ldots$ be independent $\text{Exp}(1)$ random variables. Let $W_0 = 0$.

$$\mathbb{P}[\text{Pois}(t) = 0] = e^{-t}$$
$$= \mathbb{P}[\text{Exp}(1) \geq t]$$

Connection between exponential and Poisson distributions

- Let $W_1, W_2, \ldots$ be independent $\text{Exp}(1)$ random variables. Let $W_0 = 0$.
- You should think of W_i as **waiting times** i.e. W_1 is the time you wait before the first event happens, W_2 is the time you wait between the first and second event, and so on.



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- For $t \geq 0$, let

$$N(t) = \max\{i : W_1 + \cdots + W_i \leq t\}.$$

$N(t) = 2$

Connection between exponential and Poisson distributions

a.k.a. interarrival times

- Let $W_1, W_2, \ldots$ be independent Exp(1) random variables. Let $W_0 = 0$.
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- For $t \geq 0$, let

$$N(t) = \max\{i : W_1 + \cdots + W_i \leq t\}.$$

- So, $N(t)$ denotes the number of events that happen by time t.
- In particular, $N(0) = 0$.

$N(0) = 0$

$N(W_1) = 1$

$N(W_1 + W_2) = 2$

$\vdots$

Connection between exponential and Poisson distributions

- Let $W_1, W_2, \ldots$ be independent Exp(1) random variables. Let $W_0 = 0$.
- You should think of W_i as **waiting times** i.e. W_1 is the time you wait before the first event happens, W_2 is the time you wait between the first and second event, and so on.
- For $t \geq 0$, let

$$N(t) = \max\{i : W_1 + \cdots + W_i \leq t\}.$$

- So, $N(t)$ denotes the number of events that happen by time t.
- In particular, $N(0) = 0$.
- It turns out that for all $t \geq 0$,

Randomness comes from Randomness of W_i $\longrightarrow$

$$\underbrace{N(t)}_{\text{some nonneg integer}} \sim \text{Pois}(t).$$

Connection between exponential and Poisson distributions

- For all $t \geq 0$, $N(t) \sim \text{Pois}(t)$. Why?

Connection between exponential and Poisson distributions

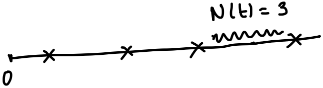
- For all $t \geq 0$, $N(t) \sim \text{Pois}(t)$. Why?
- For any $j \geq 0$,

waiting time for the first j events

waiting time for first j+1 events

$$\mathbb{P}[N(t) = j] = \mathbb{P}[W_1 + \cdots + W_j \leq t < W_1 + \cdots + W_j + W_{j+1}]$$

$j = 3$



Connection between exponential and Poisson distributions

- For all $t \geq 0$, $N(t) \sim \text{Pois}(t)$. Why?
- For any $j \geq 0$,

$W_{j+1} > t - (W_1 + \cdots + W_j)$

$\Rightarrow$

$$\mathbb{P}[N(t) = j] = \mathbb{P}[W_1 + \cdots + W_j \leq t < W_1 + \cdots + W_j + W_{j+1}]$$

$$= \int_0^t f_{W_1 + \cdots + W_j}(s) \mathbb{P}[W_{j+1} > t - s] ds$$

" $\mathbb{P}[W_1 + \cdots + W_j = s]$ " $\qquad ||$

gamma distribution $\qquad \exp(-(t-s))$

Connection between exponential and Poisson distributions

- For all $t \geq 0$, $N(t) \sim \text{Pois}(t)$. Why?
- For any $j \geq 0$,

$$\begin{aligned}\mathbb{P}[N(t) = j] &= \mathbb{P}[W_1 + \cdots + W_j \leq t < W_1 + \cdots + W_j + W_{j+1}] \\ &= \int_0^t f_{W_1+\cdots+W_j}(s)\mathbb{P}[W_{j+1} > t - s]ds \\ &= \int_0^t \left(e^{-s} \cdot \frac{s^{n-1}}{(n-1)!}\right) \cdot e^{-(t-s)}ds\end{aligned}$$

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&= \frac{e^{-t}}{(n-1)!} \int_0^t s^{n-1} ds
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&= \frac{e^{-t}}{(n-1)!}\int_0^t s^{n-1}ds \\
&= e^{-t} \cdot \frac{t^n}{n!} \\
&= \mathbb{P}[\text{Pois}(t) = j].
\end{aligned}
$$

Connection between exponential and Poisson distributions

For $t \geq 0$, let $N(t) = \max\{i : W_1 + \cdots + W_i \leq t\}$.

- We saw that $N(t) \sim \text{Pois}(t)$.

Connection between exponential and Poisson distributions

SRW:

$S_{101} - S_{100}$ is ind. of $\{S_t\}_{0 \le t \le 100}$

For $t \geq 0$, let $N(t) = \max\{i : W_1 + \cdots + W_i \leq t\}$.

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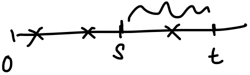
- We also have for any $0 \leq s < t$ that



$$N(t) - N(s) \text{ and } \{N(u)\}_{0 \leq u \leq s} \text{ are independent.}$$

- Why?

1. $N(t) - N(s) \sim \text{Pois}(t-s)$
2. $N(t) - N(s)$ is ind. of $\{N(u)\}_{0 \le u \le s}$.

Connection between exponential and Poisson distributions

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$$N(t) - N(s) \text{ and } \{N(u)\}_{0 \leq u \leq s} \text{ are independent.}$$

- Why? This follows from the memorylessness property of the exponential distribution.

Connection between exponential and Poisson distributions

For any $0 \leq s < t$ that

$$N(t) - N(s) \text{ and } \{N(u)\}_{0 \leq u \leq s} \text{ are independent.}$$

- Suppose $N(s) = k$ and the **arrival times** before s are $0 \leq \alpha_1 \leq \cdots \leq \alpha_k \leq s$.
- This just means that $W_1 = \alpha_1, W_1 + W_2 = \alpha_2, \ldots, W_1 + \cdots + W_k = \alpha_k$.

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- Since $N(s) = k$, we must have $W_{k+1} \geq s - \alpha_k$.

Connection between exponential and Poisson distributions

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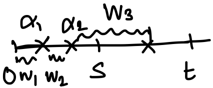
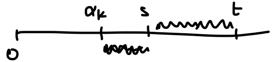


$N(S) = 2$

$$N(t) - N(s) \text{ and } \{N(u)\}_{0 \leq u \leq s} \text{ are independent.}$$



- Suppose $N(s) = k$ and the **arrival times** before s are $0 \leq \alpha_1 \leq \cdots \leq \alpha_k \leq s$.
- This just means that $W_1 = \alpha_1, W_1 + W_2 = \alpha_2, \ldots, W_1 + \cdots + W_k = \alpha_k$.
- Since $N(s) = k$, we must have $W_{k+1} > s - \alpha_k$.
- But by the memorylessness property of the exponential distribution

$$\mathbb{P}[W_{k+1} > s - \alpha_k + t \mid W_{k+1} > s - \alpha_k] = \mathbb{P}[W_{k+1} > t] = e^{-t}.$$

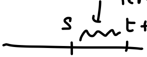
$= \mathbb{P}[\mathrm{Exp}(1) > t]$

- So, the waiting times for arrivals after s are iid Exp(1) random variables which are independent of $\{N(u)\}_{0 \leq u \leq s}$.

The Poisson Point Process

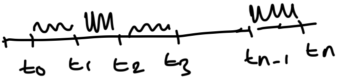
Let $\lambda > 0$. A collection of random variables $\{N(s), s \geq 0\}$ is said to be a **Poisson point process with rate** λ if

length of int is t.



- P1 $N(0) = 0$, (normalisation)
- P2 $N(t+s) - N(s) \sim \text{Pois}(\lambda t)$,
- P3 $N(t)$ has independent increments, i.e., for any $t_0 < t_1 < \cdots < t_n$,

$$N(t_1) - N(t_0), N(t_2) - N(t_1), \ldots, N(t_n) - N(t_{n-1}) \text{ are independent.}$$



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We already saw above that taking $W_0 = 0$, $W_1, W_2, \ldots$ to be iid $\text{Exp}(\lambda)$ and

$$N(s) := \max\{i : W_1 + \cdots + W_i \leq s\}$$

gives a Poisson process with rate λ.

Uniqueness of the construction

In fact, our construction of the PPP is unique.

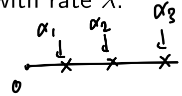
start with PPP

"Reverse our construction"

Uniqueness of the construction

In fact, our construction of the PPP is unique.

- Let $\{N(s)\}_{s \geq 0}$ be a Poisson point process with rate λ.
- Let $\alpha_0 = 0$.
- For $i \geq 1$, let $\alpha_i := \inf\{t : N(t) = i\}$.



- So, α_i is the (random) i^{th} arrival time i.e. the time that the i^{th} event happens.
- Let $W_i = \alpha_i - \alpha_{i-1}$ denote the (random) waiting time for the i^{th} event.

Uniqueness of the construction

In fact, our construction of the PPP is unique.

- Let $\{N(s)\}_{s \geq 0}$ be a Poisson point process with rate λ.
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- For $i \geq 1$, let $\alpha_i := \inf\{t : N(t) = i\}$.
- So, α_i is the (random) i^{th} arrival time i.e. the time that the i^{th} event happens.
- Let $W_i = \alpha_i - \alpha_{i-1}$ denote the (random) waiting time for the i^{th} event.
- Then, $W_1, W_2, \ldots,$ are iid $\text{Exp}(\lambda)$.

Exercise

Uniqueness of the construction

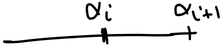
Here's the idea.

- Let's look at $W_{i+1} = \alpha_{i+1} - \alpha_i$.

Uniqueness of the construction

Here's the idea.

- Let's look at $W_{i+1} = \alpha_{i+1} - \alpha_i$.
- By conditioning on α_i, we have



$\rightarrow$ say $\alpha_i = s$

$\rightarrow$ $W_{i+1} > t$

$$\mathbb{P}[W_{i+1} > t] = \int_0^\infty \mathbb{P}[\alpha_{i+1} - s > t \mid \alpha_i = s] f_{\alpha_i}(s) ds.$$

$\Leftrightarrow$

$\alpha_{i+1} > t + s$

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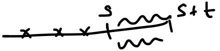
- Note that

$$\alpha_{i+1} - s > t \mid \alpha_i = s \iff N(s+t) - N(s) = 0 \mid N(s) = i.$$

$\sim \mathrm{Pois}(s+t-s) = \mathrm{Pois}(t)$

$\mathbb{P}[W_{i+1} > t] = \mathbb{P}[\mathrm{Pois}(t) = 0] = e^{-t}$



nothing happened.

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- Note that

$$\alpha_{i+1} - s > t \mid \alpha_i = s \iff N(s+t) - N(s) = 0 \mid N(s) = i.$$

- But by the independent increment property,

$$\begin{aligned}\mathbb{P}[N(s+t) - N(s) = 0 \mid N(s) = i] &= \mathbb{P}[N(s+t) - N(s) = 0] \\ &= \mathbb{P}[\text{Pois}(\lambda t) = 0] \\ &= e^{-\lambda t}\end{aligned}$$

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- As for independence, note that we actually showed that for all s,

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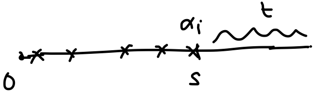

$\mathbb{P}$[this picture]

$= e^{-\lambda t}$

Uniqueness of the construction

- This shows that the waiting times $W_1, W_2, \ldots$ have $\text{Exp}(\lambda)$ distribution,
- As for independence, note that we actually showed that for all s,

$$\mathbb{P}[W_{i+1} > t \mid \alpha_i = s] = e^{-\lambda t}.$$

- Since by the independent increments property,

$$\mathbb{P}[W_{i+1} > t \mid \alpha_i = s] = \mathbb{P}[W_{i+1} > t \mid \alpha_i = s, \alpha_{i-1} = *, \ldots, \alpha_1 = *],$$

  this shows that $W_{i+1} = \alpha_{i+1} - \alpha_i$ is independent of $\alpha_1, \ldots, \alpha_i$, and hence of $W_1, \ldots, W_i$.

waiting times $\longrightarrow$ PPP

memorylessness $\longrightarrow$ ind. increments

ind. of waiting times

Infinitesimal description of the PPP

Here is another equivalent (although a bit informal) "infinitesimal" description of the Poisson point process with rate λ.

Infinitesimal description of the PPP

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same interpretation

- $N(t)$ is the number of points in $[0, t]$.
- $\mathbb{P}[\text{there is a point in } [t, t + dt]] = \lambda \cdot dt$.
- The number of points in disjoint intervals are independent.

$= \lambda\, dt +$ higher order in dt.

λ is the Rate at which events are happening

$$\underbrace{\#}_{} = \underbrace{\text{Rate}}_{\lambda} \times \underbrace{\text{time}}_{dt}$$

Infinitesimal description of the PPP

Here is another equivalent (although a bit informal) "infinitesimal" description of the Poisson point process with rate λ.

(P1) $N(0) = 0$

(P2) $N(t) - N(s) \sim \mathrm{Pois}(\lambda(t-s))$

(P3) ind. incr.

- $N(t)$ is the number of points in $[0, t]$.
- $\mathbb{P}[\text{there is a point in } [t, t+dt]] = \lambda \cdot dt$.
- The number of points in disjoint intervals are independent.

Idea: the second and third conditions, together with the Poisson approximation of the Binomial distribution show that indeed, for any $0 \le s \le t$,

$$N(t) - N(s) \sim \mathrm{Pois}(\lambda(t-s)),$$

1 n intervals of length $\left(\frac{t-s}{n}\right)$ each

s t

2 # of events in each int $\sim \mathrm{Ber}\left(\lambda\left(\frac{t-s}{n}\right)\right)$.

3 # of events b/w s & t $\sim \mathrm{Bin}\left(n, \lambda\left(\frac{t-s}{n}\right)\right) \sim \mathrm{Pois}(\lambda(t-s))$

Infinitesimal description of the PPP

Here is another equivalent (although a bit informal) "infinitesimal" description of the Poisson point process with rate λ.

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Idea: the second and third conditions, together with the Poisson approximation of the Binomial distribution show that indeed, for any $0 \leq s \leq t$,

$$N(t) - N(s) \sim \mathrm{Pois}(\lambda(t - s)),$$

and the third condition guarantees independence of increments.

The Inhomogeneous PPP

- In many situations, the condition

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  is unrealistic.

- For instance, more phone calls start during the day than in the middle of the night.

- In such cases, one can consider the more general condition

$$\mathbb{P}[\text{there is a point in } [t, t + dt]] = \lambda(t) \cdot dt.$$

The Inhomogeneous PPP

The inhomogeneous PPP may be described infinitesimally by

- $N(t)$ is the number of points in $[0, t]$.
- $\mathbb{P}$[there is a point in $[t, t + dt]] = \lambda(t)dt$.
- The number of points in disjoint intervals are independent.

3rd description for inhomogeneous PPP.

→ what about $N(t+s) - N(s)$?

→ is this also Poisson distributed.

→ indeed, it is.

$N(t+s) - N(s) \sim \text{Pois}\left(\int_s^{t+s} \lambda(u)\,du\right)$

sanity check $\lambda(t) \equiv \lambda \quad \forall t$

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By the Poisson approximation of the Binomial distribution, for any $0 \leq s \leq t$,

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Why?

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Why?

$$\mathbb{P}[N(t) - N(s) = k] \approx \mathbb{P}\left[\text{Ber}\,\frac{\lambda(s)}{n} + \text{Ber}\,\frac{\lambda(s+n^{-1})}{n} + \cdots + \text{Ber}\,\frac{\lambda(t-n^{-1})}{n} = k\right]$$

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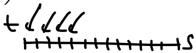

Why?

$$\begin{aligned}
\mathbb{P}[N(t) - N(s) = k] &\approx \mathbb{P}\left[\text{Ber}\,\frac{\lambda(s)}{n} + \text{Ber}\,\frac{\lambda(s+n^{-1})}{n} + \cdots + \text{Ber}\,\frac{\lambda(t-n^{-1})}{n} = k\right] \\
&\approx \mathbb{P}\left[\text{Pois}\left(\underbrace{\frac{\lambda(s)}{n} + \frac{\lambda(s+n^{-1})}{n} + \frac{\lambda(t-n^{-1})}{n}}_{\approx \int_s^t \lambda(u)\,du}\right) = k\right]
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$$\begin{aligned}
\mathbb{P}[N(t) - N(s) = k] &\approx \mathbb{P}\left[\text{Ber}\,\frac{\lambda(s)}{n} + \text{Ber}\,\frac{\lambda(s + n^{-1})}{n} + \cdots + \text{Ber}\,\frac{\lambda(t - n^{-1})}{n} = k\right] \\
&\approx \mathbb{P}\left[\text{Pois}\left(\frac{\lambda(s)}{n} + \frac{\lambda(s + n^{-1})}{n} + \frac{\lambda(t - n^{-1})}{n}\right) = k\right] \\
&\approx \mathbb{P}\left[\text{Pois}\left(\int_s^t \lambda(u)du\right) = k\right].
\end{aligned}$$

Construction of inhomogeneous PPP

Let

$$\Lambda(s,t) := \int_s^t \lambda(u)du. \qquad \Lambda(0,t) = \int_0^t \lambda(u)du$$

- $N(t) = N^{\text{hom}}(\Lambda(0,t))$ is an inhomogeneous PPP, where $N^{\text{hom}}(\cdot)$ is a (homogeneous) PPP with rate 1. Why?

$*$ $N(0) = 0$

$(*)$ ind inc. is clear (b/c N^{hom} has ind. inc.)

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  Clearly $N(0) = 0$ and $N(t)$ has independent increments. As for the distribution, note that

$$N(t+dt) - N(t) = N^{\text{hom}}(\Lambda(0,t+dt)) - N^{\text{hom}}(\Lambda(0,t))$$

$$\approx \Lambda(t) + \lambda(t)dt$$

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$N^{hom}(s+ds) - N^{hom}(s) = ds$

$\approx \quad \lambda(t)dt = \{\Lambda(0,t) + \lambda(t)dt - \Lambda(0,t)\}.$

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