STATS 217: Introduction to Stochastic Processes I

Lecture 7

Recap

The **inhomogeneous PPP** may be described infinitesimally by

- $N(t)$ is the number of points in $[0, t]$.
- $\mathbb{P}[\text{there is a point in } [t, t + dt]] = \lambda(t)dt$ i.e.,
  - $\mathbb{P}[N(t+\epsilon) - N(t) = 0] = 1 - \lambda\epsilon + o(\epsilon)$. complementary case: no point
  - $\mathbb{P}[N(t+\epsilon) - N(t) = 1] = \lambda\epsilon + o(\epsilon)$. exactly one point
  - $\mathbb{P}[N(t+\epsilon) - N(t) > 1] = o(\epsilon)$, more than one point

  where $o(\epsilon)/\epsilon \to 0$ as $\epsilon \to 0$.
- The number of points in disjoint intervals are independent.

mean of. thus poisson

We saw that for any $0 \leq s \leq t$, $N(t) - N(s) \sim \text{Pois}\left(\int_s^t \lambda(u)du\right)$.

Construction: $N(t) = N^{\text{hom}}(\Lambda(0,t))$ is an inhomogeneous PPP, where $N^{\text{hom}}(\cdot)$ is a (homogeneous) PPP with rate 1. $\Lambda(0,t) = \int_0^t \lambda(u)\,du$

Homogeneous case: Taking $\lambda(t) \equiv \lambda$ gives a (homogeneous) PPP of rate λ, in which case the waiting (interarrival) times are i.i.d. $\text{Exp}(\lambda)$ (not true in the general inhomogeneous case).

Superposition of Poisson processes

Let $N_1(t), \ldots, N_k(t)$ be independent Poisson processes with rates $\lambda_1, \ldots, \lambda_k$.

- Then, $N_1(t) + \cdots + N_k(t)$ is a Poisson process with rate $\lambda_1 + \cdots + \lambda_k$.
- Why?

Superposition of Poisson processes

(P1) : $N(0) = 0$ (P2) : $N(t) - N(s) \sim \text{Poi}(\lambda(t-s))$
(P3) : ind. of increments

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- Why? Recall properties (P1), (P2), (P3) from last lecture.
- (P1) and (P3) are immediate.

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- (P1) and (P3) are immediate.
- As for (P2), for any $s \leq t$

$$\begin{aligned} & N_1(t) + \cdots + N_k(t) - (N_1(s) + \cdots + N_k(s)) \\ & = (N_1(t) - N_1(s)) + \cdots + (N_k(t) - N_k(s)) \\ & \sim \text{Pois}(\lambda_1(t-s)) + \cdots + \text{Pois}(\lambda_k(t-s)) \end{aligned}$$

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& = (N_1(t) - N_1(s)) + \cdots + (N_k(t) - N_k(s)) \\
& \sim \mathrm{Pois}(\lambda_1(t-s)) + \cdots + \mathrm{Pois}(\lambda_k(t-s)) \\
& \sim \mathrm{Pois}((\lambda_1 + \cdots + \lambda_k)(t-s)).
\end{aligned}
$$

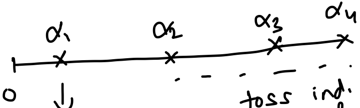
Poisson thinning

- Let $\{N(s)\}_{s \geq 0}$ denote a Poisson process with rate λ.
- Let $\alpha_1, \alpha_2, \ldots$ denote the (random) "arrival times".
- Let $Y_1, Y_2, \ldots$ denote a sequence of iid random variables.
- For each $j \in \text{supp}(Y_1)$, let $p_j = \mathbb{P}[Y_1 = j]$
- For each $j \in \text{supp}(Y_1)$, define

$$N_j(s) := |\{i \in \{1, 2, \ldots, \lfloor N(s) \rfloor\} : Y_i = j\}|.$$

$N(t)$

$N_{blue}(t) + N_{Red}(t)$



toss a coin
heads : blue
tails : Red

toss ind. coin for each of these

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clear from infinitesimal description.

- Then,
  - $\{N_j(s)\}_{s\geq 0}$ is a Poisson process with rate $p_j\lambda$ and
  - $\{N_1(s)\}_{s\geq 0}, \{N_2(s)\}_{s\geq 0}, \ldots$ are independent.

$$\underline{\mathbb{P}}\left(N_1(t_1) = *,\ N_1(t_2) = * \ \ldots\ N_1(t_k) = * \ \middle|\ \underline{N}_2(s_1) = *, \ldots, N_2(s_\ell) = * \right)$$

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- Why? Can check by direct calculation, but for intuition, consider the case when $Y \sim \text{Ber}(p)$.

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- For independence, note that

wts that this is $(1-p)\lambda\,dt$

$$\mathbb{P}[N_0 \text{ has a point in} [t, t+dt] \mid N_1 \text{ has a point in} [t, t+dt]]$$

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$$\mathbb{P}[N_0 \text{ has a point in}[t, t+dt] \mid N_1 \text{ has a point in}[t, t+dt]]$$
$$= \frac{\mathbb{P}[N_0 \text{ and } N_1 \text{ have points in } [t, t+dt]]}{p\lambda dt}$$

$\longrightarrow \mathbb{P}[N_1$ has a point in $[t, t+dt]]$.

in order to compute this to first order in dt, want numerator up to $(dt)^2 + \cdots$

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\begin{aligned}
&\mathbb{P}[N_0 \text{ has a point in}[t, t+dt] \mid N_1 \text{ has a point in}[t, t+dt]] \\
&= \frac{\mathbb{P}[N_0 \text{ and } N_1 \text{ have points in } [t, t+dt]]}{p\lambda dt} \\
&\approx (p\lambda dt)^{-1} \cdot \left(\underbrace{\boxed{e^{-\lambda dt} \cdot \frac{(\lambda dt)^2}{2}}\left(p(1-p) + (1-p)p\right) + o((\lambda dt)^2)}\right)
\end{aligned}
$$

$\mathbb{P}$ [$N(t)$ has 2 points in $[t, t+dt]$]

$= \mathbb{P}[\text{Poi}(\lambda dt) = 2] = e^{-\lambda dt}\dfrac{(\lambda dt)^2}{2!}$

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&\approx \cancel{(p\lambda dt)}^{-1} \cdot \left(\underbrace{e^{-\lambda dt}}_{\approx 1} \cdot \frac{(\lambda dt)^2}{\cancel{2}} \cdot \underbrace{(p(1-p) + (1-p)p)}_{\cancel{2p}(1-p)} + o((\lambda dt)^2) \right) \\
&\approx (1-p)\lambda dt \\
&= \mathbb{P}[N_0 \text{ has a point in } [t, t+dt]].
\end{aligned}
$$

$\mathbb{P}$[one red & one green in $[t, t+dt]$]

$= 2p(1-p) \cdot e^{-\lambda dt} \dfrac{(\lambda dt)^2}{2!}$

Poisson thinning

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- Let $\{N(s)\}_{s\geq 0}$ denote an inhomogeneous PPP with rate $\lambda(s)$.
- Let $\{Y(s)\}_{s\geq 0}$ denote a collection of independent random variables, each with support $\{1,\ldots,k\}$.
- Let $\alpha_1, \alpha_2, \ldots$ denote the random arrival times.
- For $j = 1,\ldots,k$, define

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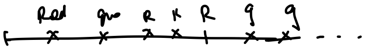
s $s+ds$

$\lambda(s)\,ds$

$\mathbb{P}[Y(s)=j]$

- Then,
  - $N_j(s)$ is an inhomogeneous PPP with rate $\lambda(s)\mathbb{P}[Y(s)=j]$.
  - $\{N_1(s)\}_{s\geq 0},\dots,\{N_k(s)\}_{s\geq 0}$ are independent processes

Example



Example 2.5 from Durrett. Given a Poisson process of red arrivals with rate λ and an independent Poisson process of green arrivals with rate μ, what is the probability that we will get 6 red arrivals before a total of 4 green ones?

idea: view Red & green PPP as thinned versions of a common PPP.

detail: $N(t)$ has rate $(\lambda+\mu)$

$\mathbb{P}[\text{Red}] = \frac{\lambda}{\lambda+\mu}$ $\quad$ $\mathbb{P}[\text{green}] = \frac{\mu}{\lambda+\mu}$

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step 1 : "merge" via thinning.

- Equivalently, at least 6 red arrivals in the first 9.

in 9 coin tosses

$$P_{head} = \frac{\lambda}{\lambda+\mu}$$

$$\mathbb{P}[\geq 6 \text{ heads}] = ?$$

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- Equivalently, at least 6 red arrivals in the first 9.
- By thinning,

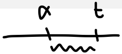
$$\sum_{k=6}^{9} \left(\frac{\lambda}{\lambda+\mu}\right)^{k} \left(\frac{\mu}{\lambda+\mu}\right)^{9-k} \binom{9}{k}.$$

$p_{red} \propto \lambda$

$p_{green} \propto \mu$

$p_{red} + p_{green} = 1$

Example



$\mathbb{P}$ [call lasts for time $\leq a$] $= G(a)$ → CDF of dis. of call lengths.

Example 2.4 from Durrett. Consider a model of telephone traffic in which the system starts empty at time 0. Suppose that the starting times of the calls is a Poisson process with rate λ and that the probability a call started at time s has ended by time t is $G(t-s)$, where G is some CDF with $G(0) = 0$ and mean μ. What is the distribution of the number of calls still in progress at time t? calls are ind.

- Call starting at time $\alpha \in [0, t]$ is kept with probability $(1 - G(t - \alpha))$.
- Therefore, by thinning, number of calls in progress at time t is Poisson with mean

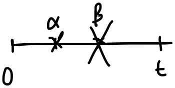
change of vars

$$\int_{s=0}^{t} \lambda(1 - G(t-s))ds = \lambda \int_{r=0}^{t} (1 - G(r))dr.$$

- Let $t \to \infty$ to see that in the long run, the number of calls in the system is Poisson with mean

$$\lambda \int_{r=0}^{\infty} \mathbb{P}(G \geq r)dr = \lambda\mu.$$

→ how can I have a call in my sys at time t?



prob of this $(1 - G(t-\alpha))$

- it must start at $\alpha \in [0,t]$
- must last for $\geq (t-\alpha)$

Compound Poisson processes

Each of the 'thinned' Poisson processes is a special case of a **compound Poisson process**.

- Let $\{N(s)\}_{s \geq 0}$ denote a Poisson process with rate λ.
- Let $\alpha_1, \alpha_2, \ldots$ denote the (random) "arrival times".
- Let $Y_1, Y_2, \ldots$ denote a sequence of iid random variables. Let $Y_0 = 0$.
- Let

$$S(t) = Y_0 + Y_1 + \cdots + Y_{N(t)}.$$

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random sum w/ Random # of summands.

- Then, $\mathbb{E}[S(t)] = \mathbb{E}[Y_1] \cdot \mathbb{E}[N(t)]$ by the same argument as for branching processes.

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- Then, $\mathbb{E}[S(t)] = \mathbb{E}[Y_1] \cdot \mathbb{E}[N(t)]$ by the same argument as for branching processes.
- Also, by the same argument as on this week's homework,

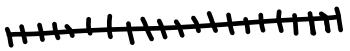
$$\mathrm{Var}[S(t)] = \mathbb{E}[N(t)] \cdot \mathrm{Var}(Y_1) + \mathrm{Var}[N(t)] \cdot \mathbb{E}[Y_1]^2.$$

$S(t) = Z_n \quad N(t) = Z_{n-1} \quad Y_1 = \xi$

Poisson conditioning

homogeneous case

- Let $\{N(s)\}_{s \geq 0}$ be a Poisson process with rate λ.
- Let $\alpha_1, \alpha_2, \ldots,$ denote the (random) "arrival times".
- Conditioned on $N(t) = n$, what is the distribution of $\alpha_1, \ldots, \alpha_n$?



there are 100 points
but what's the distribution?

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- It turns out that

$$\{\alpha_1, \ldots, \alpha_n\} \sim \{u_1, \ldots, u_n\},$$

  where $u_1, \ldots, u_n$ are iid uniformly distributed in $[0, t]$.

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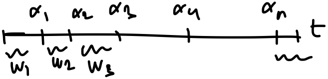
- Why? Again, this is intuitive from the infinitesimal description of the process.

Poisson conditioning

Formally,



$$\mathbb{P}[\text{arrival times } \alpha_1, \ldots, \alpha_n \mid N(t) = n]$$
$$= \mathbb{P}[N(t) = n]^{-1} \mathbb{P}[W_1 = \alpha_1, \ldots, W_n = \alpha_n - \alpha_{n-1}, W_{n+1} > t - \alpha_n]$$

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&= \mathbb{P}[N(t) = n]^{-1} \lambda e^{-\lambda\alpha_1} \cdot \ldots \lambda e^{-\lambda(\alpha_n - \alpha_{n-1})} \cdot e^{-\lambda(t-\alpha_n)}
\end{aligned}
$$

$$\lambda e^{-\lambda \cancel{\alpha_1}} \lambda e^{-\lambda \cancel{(\alpha_2 - \alpha_1)}}$$

Poisson conditioning

$$\mathbb{P}(N(t)=n) = \mathbb{P}[\text{Poi}(\lambda t)=n]$$

$$= \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

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&= \mathbb{P}[N(t) = n]^{-1}\lambda e^{-\lambda\alpha_1} \cdot \ldots \lambda e^{-\lambda(\alpha_n - \alpha_{n-1})} \cdot e^{-\lambda(t-\alpha_n)} \\
&= \mathbb{P}[N(t) = n]^{-1} \cdot \lambda^n e^{-\lambda t},
\end{aligned}
$$

which does not depend on $\alpha_1, \ldots, \alpha_n$.

distribution of n iid uniform points in $[0,t]$

$(\alpha_1 \ldots \alpha_n) \in [0,t]^n$

unif + iid $\Leftrightarrow$ density is $\left(\frac{1}{t}\right)^n$. $n!$ factor from permutations

Example: simulating a PPP

Here is a practical application of Poisson conditioning.

- How might one generate (on a computer) a PPP with rate λ in the time interval $[0, t]$?

$$N(t) - N(0) \sim \text{Poi}(\lambda t)$$

Example: simulating a PPP

Here is a practical application of Poisson conditioning.

- How might one generate (on a computer) a PPP with rate λ in the time interval $[0, t]$?
- Poisson conditioning shows that we can do this in two easy steps.
  - First, generate $N(t) \sim \text{Pois}(\lambda t)$.
  - Next, generate $\alpha_1, \ldots, \alpha_{N(t)}$ iid uniformly in $[0, t]$.

can also do this using
i.i.d exp. waiting times