STATS 217: Introduction to Stochastic Processes I

Lecture 8

Random variables and stochastic processes

- A **random variable** is a function

$$X : \Omega \to \mathbb{R},$$

where Ω is a probability space (think of this as the space of outcomes of a random experiment).

Random variables and stochastic processes

- A **random variable** is a function

$$X : \Omega \to \mathbb{R},$$

where Ω is a probability space (think of this as the space of outcomes of a random experiment).

- A **stochastic process** is a collection of random variables

$$(X_t)_{t \in \mathcal{T}}.$$

Random variables and stochastic processes

- A **random variable** is a function

$$X : \Omega \to \mathbb{R},$$

where Ω is a probability space (think of this as the space of outcomes of a random experiment).

- A **stochastic process** is a collection of random variables

$$(X_t)_{t \in \mathcal{T}}.$$

- The most common choices for us will be

$$\mathcal{T} = \mathbb{Z}^{\geq 0} = \{0, 1, 2, \ldots, \},$$
$$\mathcal{T} = \mathbb{Z},$$
$$\mathcal{T} = \mathbb{R}. \quad \text{OR} \quad \mathcal{T} = \mathbb{R}^{\geq 0}$$

Markov chains

$X_0, X_1, X_2, \ldots$

starting state

- A **discrete time Markov chain (DTMC)** is a stochastic process $(X_t)_{t \in \mathbb{Z}^{\geq 0}}$ satisfying the **Markov property**

$Y_{n-1} = x_{n-1}$

$$\mathbb{P}[X_{n+1} = x_{n+1} \mid X_n = x_n, \ldots, X_0 = x_0] = \mathbb{P}[X_{n+1} = x_{n+1} \mid X_n = x_n]$$

for all $n \geq 1$ and $x_0, \ldots, x_{n+1}$.

- In other words, conditioned on the present, the future is independent of the past.

Markov chains

think of finite state random vars

- A **discrete time Markov chain (DTMC)** is a stochastic process $(X_t)_{t \in \mathbb{Z}^{\geq 0}}$ satisfying the **Markov property**

$$\mathbb{P}[X_{n+1} = x_{n+1} \mid X_n = x_n, \ldots, X_0 = x_0] = \mathbb{P}[X_{n+1} = x_{n+1} \mid X_n = x_n]$$

  for all $n \geq 1$ and $x_0, \ldots, x_{n+1}$.
- In other words, conditioned on the present, the future is independent of the past.
- A DTMC is **time homogeneous** if

$$\mathbb{P}[X_{n+1} = j \mid X_n = i] = \mathbb{P}[X_{m+1} = j \mid X_m = i]$$

  for all i, j and all times n, m.

Markov chains

- A **discrete time Markov chain (DTMC)** is a stochastic process $(X_t)_{t \in \mathbb{Z}^{\geq 0}}$ satisfying the **Markov property**

$$\mathbb{P}[X_{n+1} = x_{n+1} \mid X_n = x_n, \ldots, X_0 = x_0] = \mathbb{P}[X_{n+1} = x_{n+1} \mid X_n = x_n]$$

  for all $n \geq 1$ and $x_0, \ldots, x_{n+1}$.
- In other words, conditioned on the present, the future is independent of the past.
- A DTMC is **time homogeneous** if

$$\mathbb{P}[X_{n+1} = j \mid X_n = i] = \mathbb{P}[X_{m+1} = j \mid X_m = i]$$

  for all i, j and all times n, m.
- From now on, unless specified otherwise, a DTMC is assumed to be time homogeneous.

Transition matrix

A DTMC is completely specified by the following pieces of information.

- The **state space** S, which is the collection of all possible values that $X_0, X_1, \ldots,$ could take.
- The **initial state** X_0.
- The **transition probabilities**

$$p_{ij} := \mathbb{P}[X_{n+1} = j \mid X_n = i] \quad \forall i, j \in S.$$

in general: p_{ij} depends on $\underline{n}$

but in the time hom. case, it does not.

Transition matrix

A DTMC is completely specified by the following pieces of information.

- The **state space** S, which is the collection of all possible values that $X_0, X_1, \ldots,$ could take.
- The **initial state** X_0.
- The **transition probabilities**

$$p_{ij} := \mathbb{P}[X_{n+1} = j \mid X_n = i] \quad \forall i, j \in S.$$

- By time homogeneity, the right hand side depends only on i, j and not on n.

Transition matrix

A DTMC is completely specified by the following pieces of information.

- The **state space** S, which is the collection of all possible values that $X_0, X_1, \ldots,$ could take.
- The **initial state** X_0.
- The **transition probabilities**

$$p_{ij} := \mathbb{P}[X_{n+1} = j \mid X_n = i] \quad \forall i, j \in S.$$

- By time homogeneity, the right hand side depends only on i, j and not on n.
- It will be useful to combine all of the transition probabilities into an $|S| \times |S|$ **transition matrix** P,

$$(P)_{ij} := p_{ij}.$$

Transition matrix

A DTMC is completely specified by the following pieces of information.

- The **state space** S, which is the collection of all possible values that $X_0, X_1, \ldots,$ could take.
- The **initial state** X_0.
- The **transition probabilities**

$$p_{ij} := \mathbb{P}[X_{n+1} = j \mid X_n = i] \quad \forall i, j \in S.$$

- By time homogeneity, the right hand side depends only on i, j and not on n.
- It will be useful to combine all of the transition probabilities into an $|S| \times |S|$ **transition matrix** P,

$$(P)_{ij} := p_{ij}.$$

p_{ij} = prob. of jumping from i to j

- Note that P is **row-stochastic** i.e.

$$\sum_{j \in S} p_{ij} = 1 \quad \forall i \in S.$$

$p_{ij} \geq 0.$

Examples

Two state Markov chain.

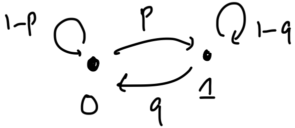
- State space: $S = \{0, 1\}$.
- Transition matrix:

$$\begin{array}{c c} & \begin{array}{cc} 0 & \quad 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} 1-p & p \\ q & 1-q, \end{pmatrix} \end{array}$$

  for some $p, q \in [0, 1]$.

- $X_0 = 0$

what is this doing?



at every step, toss a coin (ind.)

$\mathbb{P}(\text{heads}) = p$

Examples

$P_{i,i+1} = \frac{1}{2}$
$P_{i,i-1} = \frac{1}{2}$ $\forall i$ everything else is 0.

Symmetric simple random walk on the integers.

doubly inf. matrix

$\mathbb{Z}$

$$\begin{pmatrix} 0 & \frac{1}{2} & \cdots & \\ \frac{1}{2} & \ddots & & \frac{1}{2} \\ & & \frac{1}{2} & 0 \end{pmatrix}$$

- What is the state space?
- What are the transition probabilities?

Examples

Gambler's ruin stopped at -\$100 or \$200.

- What is the state space? $-100, -99, \ldots, 200$
- What are the transition probabilities?

$$p_{i,i+1} = 1/2 \qquad p_{i,i-1} = 1/2$$

AND. $p_{200,200} = 1$

$p_{-100,-100} = 1$

Examples

Gambler's ruin stopped at -\$100 or \$200.

- What is the state space?
- What are the transition probabilities?

- A state $i \in S$ is called **absorbing** if $p_{ii} = 1$.
- What are the absorbing states, if any?

Examples

Branching process with $Z_0 = 1$ and offspring distribution ξ.

nonneg integers

- What is the state space?
- What are the transition probabilities?
- What are the absorbing states, if any?

$$= \mathbb{P}[Z_{n+1} = j \mid Z_n = i]$$

$$= \mathbb{P}\Big[\underbrace{\sum_{k=1}^{i} \xi_k = j}_{\text{some func. of only } i \text{ and } j}\Big].$$

0 (always)

if, say, $\mathbb{P}[\xi = 1] = \underline{1}$, then $\underline{1}$ also absorbing.

Examples

Coupon collector. There are n different types of coupons, say, $\{1, \ldots, n\}$. Each day you get a uniformly random coupon (repetitions allowed). You stop once you've collected all n types of coupons.

Examples

Coupon collector. There are n different types of coupons, say, $\{1, \ldots, n\}$. Each day you get a uniformly random coupon (repetitions allowed). You stop once you've collected all n types of coupons.

Let X_i denote the number of different types of coupons you've collected by the end of day i. You start with $X_0 = 0$ coupons.

Examples

Coupon collector. There are n different types of coupons, say, $\{1, \dots, n\}$. Each day you get a uniformly random coupon (repetitions allowed). You stop once you've collected all n types of coupons.

Let X_i denote the number of different types of coupons you've collected by the end of day i. You start with $X_0 = 0$ coupons.

- What is the state space? → $0, 1, 2, \dots, n$
- What are the transition probabilities? → $p_{ij} = ?$

$$\begin{cases} p_{i,i+1} = \frac{n-i}{n} \\ p_{i,i} \\ p_{n,n} = 1 \text{ (absorbing)} \end{cases}$$

everything else 0.

Examples

Coupon collector. There are n different types of coupons, say, $\{1, \ldots, n\}$. Each day you get a uniformly random coupon (repetitions allowed). You stop once you've collected all n types of coupons.

Let X_i denote the number of different types of coupons you've collected by the end of day i. You start with $X_0 = 0$ coupons.

- What is the state space?
- What are the transition probabilities?
- Are there any absorbing states?

Examples

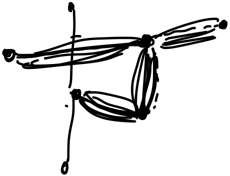
Coupon collector. There are n different types of coupons, say, $\{1, \ldots, n\}$. Each day you get a uniformly random coupon (repetitions allowed). You stop once you've collected all n types of coupons.

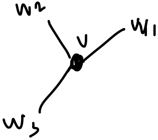
Let X_i denote the number of different types of coupons you've collected by the end of day i. You start with $X_0 = 0$ coupons.

- What is the state space?
- What are the transition probabilities?
- Are there any absorbing states?
- On the problem set, you will study the time it takes to collect all n types of coupons.

Examples

Random walk on a graph. Let $G = (V, E)$ be an undirected graph on vertices $V = \{1, \ldots, n\}$ and edges E. We start at the vertex v_0 and at every time, move to a uniformly random neighbor of the current vertex.





Examples

Random walk on a graph. Let $G = (V, E)$ be an undirected graph on vertices $V = \{1, \ldots, n\}$ and edges E. We start at the vertex v_0 and at every time, move to a uniformly random neighbor of the current vertex.

Let X_i denote our position at time i.

Examples

of neighbors of v = "degree of v"
= $\deg_G(v)$

Random walk on a graph. Let $G = (V, E)$ be an undirected graph on vertices $V = \{1, \ldots, n\}$ and edges E. We start at the vertex v_0 and at every time, move to a uniformly random neighbor of the current vertex.

Let X_i denote our position at time i.

$w \sim v$

w and v are conn. by an edge.

- What is the state space? → V
- What are the transition probabilities?

$$P_{vw} = \begin{cases} 0 & \text{if } w \nsim v \\ \frac{1}{\deg_G(v)} & \text{else} \end{cases}$$

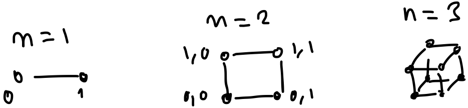
Examples

Random walk on a graph. Let $G = (V, E)$ be an undirected graph on vertices $V = \{1, \ldots, n\}$ and edges E. We start at the vertex v_0 and at every time, move to a uniformly random neighbor of the current vertex.

Let X_i denote our position at time i.

- What is the state space?
- What are the transition probabilities?
- Are there any absorbing states?

yes, if there "isolated" vtcs.



Simple random walk on the ***n*****-dimensional hypercube**. The *n*-dimensional hypercube is the undirected graph on $V = \{0,1\}^n$ where $u, v \in V$ are connected by an edge $e \in E$ if and only if u and v differ in exactly one coordinate.

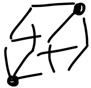
Examples

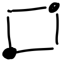
Simple random walk on the n-dimensional hypercube. The n-dimensional hypercube is the undirected graph on $V = \{0,1\}^n$ where $u, v \in V$ are connected by an edge $e \in E$ if and only if u and v differ in exactly one coordinate.

- What is the state space?
- What are the transition probabilities?

Examples

Simple random walk on the *n*-dimensional hypercube. The *n*-dimensional hypercube is the undirected graph on $V = \{0,1\}^n$ where $u, v \in V$ are connected by an edge $e \in E$ if and only if u and v differ in exactly one coordinate.





- What is the state space?
- What are the transition probabilities?
- Starting from $(0,0,\ldots,0)$, can the random walk hit $(1,1,\ldots,1)$ in an even number of steps?

$\Leftarrow$) n is even.

$X_0 = 0$

$n = 2$.

at even times:
either $(0,0)$ or $(1\ 1)$

Examples

Lazy random walk on the *n*-dimensional hypercube. Transition matrix

$$P_{\text{lazy}} = \frac{1}{2}I + \frac{1}{2}P_{\text{simple}},$$

$$P'_{\text{lazy}} = \frac{3}{4} I + \frac{1}{4} P_{\text{simple}}$$

where I is the $2^n \times 2^n$ identity matrix and P_{simple} is the transition matrix of the simple random walk on the n-dimensional hypercube.

one way to interpret the formula:

- What is this Markov chain doing?

H T

do nothing

$$P' = \frac{3}{4} I + \frac{1}{2} P_{\text{simple}}.$$

$$\sum_j P'_{ij} = \frac{3}{4}\sum_j I_{ij} + \frac{1}{2}\sum_j (P_{\text{simple}})_{ij}.$$

Examples

Lazy random walk on the n-dimensional hypercube. Transition matrix

$$P_{\text{lazy}} = \frac{1}{2} I + \frac{1}{2} P_{\text{simple}},$$

where I is the $2^n \times 2^n$ identity matrix and P_{simple} is the transition matrix of the simple random walk on the n-dimensional hypercube.

- What is this Markov chain doing?
- Starting at $(0, 0, \ldots, 0)$, can the random walk hit $(1, 1, \ldots, 1)$ in an even number of steps?

Examples

The Ehrenfest urn. n balls are distributed among two urns, urn A and urn B. At each time, we select a ball uniformly at random and move it from its current urn to the other urn.

- How can we model this as a Markov chain?

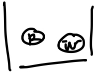
X_i is # of balls in A

$P_{i,i+1}$

$P_{i,i-1} = \frac{i}{n}$

Examples

Polya's urn. We start with a single urn containing a red ball and a white ball. At each time, we select a ball uniformly at random and return it to the urn along with a new ball of the same color.



- How can we model this as a Markov chain?

(Red balls, white balls)

$X_0 = (1, 1)$

$$i, j \xrightarrow{\frac{i}{i+j}} i+1, j$$

$$i, j \xrightarrow[\frac{j}{i+j}]{} i, j+1$$

Examples

Polya's urn. We start with a single urn containing a red ball and a white ball. At each time, we select a ball uniformly at random and return it to the urn along with a new ball of the same color.

- How can we model this as a Markov chain?
- Let R_k denote the number of red balls in the urn after k new balls have been added. What are the possible values that R_k can take?

$\{1, 2, \ldots, k+1\}$

Examples

Polya's urn. We start with a single urn containing a red ball and a white ball. At each time, we select a ball uniformly at random and return it to the urn along with a new ball of the same color.

- How can we model this as a Markov chain?
- Let R_k denote the number of red balls in the urn after k new balls have been added. What are the possible values that R_k can take?
- On the homework, you will find the distribution of R_k.

Examples

Free throws. Consider a basketball player who makes free throws with the following probabilities

1/2 if she missed the last two times

2/3 if she made one of the last two throws

3/4 if she made both of her last two throws.

→ yes, by tracking 2 most recent throws

- Can this be modelled as a Markov chain?

H : hit

M : miss

$$
\begin{array}{cc}
 & \begin{array}{cccc} HH & HM & MH & MM \end{array} \\
\begin{array}{c} HH \\ HM \\ MH \\ MM \end{array} & \begin{pmatrix} 3/4 & 1/4 & 0 & 0 \\ & & & \\ & & & \\ 0 & 0 & * & * \end{pmatrix}
\end{array}
$$

n is M

n+1 is H

n−1 is H

n is M

Examples

let's say in a game, you have 10 attempts already

what happens for # 11?

Free throws. Consider a basketball player who makes free throws with the following probabilities

$x_1, x_2, \ldots, x_{10}, x_{11}$

1/2 if she missed the last two times $P[x_{11} = H \mid x_1, \ldots, x_{10}]$

2/3 if she made one of the last two throws

3/4 if she made both of her last two throws.

- Can this be modelled as a Markov chain?
- What is the state space?
- What are the transition probabilities?

Multi-step transition probabilities

- The transition probability p_{ij} tells us the probability of going from i to j in *one* step, i.e.

$$p_{ij} = \mathbb{P}[X_1 = j \mid X_0 = i].$$

Multi-step transition probabilities

- The transition probability p_{ij} tells us the probability of going from i to j in *one* step, i.e.

$$p_{ij} = \mathbb{P}[X_1 = j \mid X_0 = i].$$

- What about the probability of going from i to j in *two* steps i.e. what is

$$p_{ij}^2 := \mathbb{P}[X_2 = j \mid X_0 = i]?$$

Multi-step transition probabilities

- The transition probability p_{ij} tells us the probability of going from i to j in *one* step, i.e.

$$p_{ij} = \mathbb{P}[X_1 = j \mid X_0 = i].$$

- What about the probability of going from i to j in *two* steps i.e. what is

$$p_{ij}^2 := \mathbb{P}[X_2 = j \mid X_0 = i]?$$

- Well, to go from i to j in two steps, we must go from i to some state $k \in \Omega$ in one step and then from k to j in one step.

Multi-step transition probabilities

Using the law of total probability, we have

$$\mathbb{P}[X_2 = j \mid X_0 = i] = \sum_{k \in \Omega} \mathbb{P}[X_1 = k \wedge X_2 = j \mid X_0 = i]$$

Multi-step transition probabilities

Using the law of total probability, we have

$$\begin{aligned}\mathbb{P}[X_2 = j \mid X_0 = i] &= \sum_{k \in \Omega} \mathbb{P}[X_1 = k \wedge X_2 = j \mid X_0 = i] \\ &= \sum_{k \in \Omega} \mathbb{P}[X_1 = k \mid X_0 = i] \mathbb{P}[X_2 = j \mid X_0 = i \wedge X_1 = k]\end{aligned}$$

Multi-step transition probabilities

Using the law of total probability, we have

$$
\begin{aligned}
\mathbb{P}[X_2 = j \mid X_0 = i] &= \sum_{k \in \Omega} \mathbb{P}[X_1 = k \wedge X_2 = j \mid X_0 = i] \\
&= \sum_{k \in \Omega} \mathbb{P}[X_1 = k \mid X_0 = i] \mathbb{P}[X_2 = j \mid X_0 = i \wedge X_1 = k] \\
&= \sum_{k \in \Omega} \mathbb{P}[X_1 = k \mid X_0 = i] \mathbb{P}[X_2 = j \mid X_1 = k]
\end{aligned}
$$

Multi-step transition probabilities

Using the law of total probability, we have

$$\begin{aligned}
\mathbb{P}[X_2 = j \mid X_0 = i] &= \sum_{k \in \Omega} \mathbb{P}[X_1 = k \wedge X_2 = j \mid X_0 = i] \\
&= \sum_{k \in \Omega} \mathbb{P}[X_1 = k \mid X_0 = i] \mathbb{P}[X_2 = j \mid X_0 = i \wedge X_1 = k] \\
&= \sum_{k \in \Omega} \mathbb{P}[X_1 = k \mid X_0 = i] \mathbb{P}[X_2 = j \mid X_1 = k] \\
&= \sum_{k \in \Omega} p_{ik} p_{kj} \\
&= (P^2)_{ij}.
\end{aligned}$$

Multi-step transition probabilities

- There is nothing special about two steps here and you should check that the same argument gives

$$p_{ij}^n := \mathbb{P}[X_n = j \mid X_0 = i] = (P^n)_{ij} \quad \forall n \geq 1.$$

Multi-step transition probabilities

- There is nothing special about two steps here and you should check that the same argument gives

$$p_{ij}^{n} := \mathbb{P}[X_n = j \mid X_0 = i] = (P^n)_{ij} \quad \forall n \geq 1.$$

- Since for any non-negative integers ℓ, m,

$$P^{\ell+m} = P^{\ell} P^{m},$$

we obtain the **Chapman-Kolmogorov equations**

$$p_{ij}^{\ell+m} = \sum_{k \in \Omega} p_{ik}^{\ell} p_{kj}^{m}.$$